MTH 3318 Test #1 - Solutions

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Instructions. Fully document your work.

For problems 1 - 2, prove one using Mathematical Induction:

1. For $0 \leq a \leq b$; prove that $a^n \leq b^n$.

Proof.

Step #1: Show true for $n = 1$.

$$a^1 = \underbrace{a \leq b}_{\text{given}} = b^1$$

i.e., $a^1 \leq b^1$ True.

Step #2: Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $a^k \leq b^k$ for some natural number k, and show that $a^{k+1} \leq b^{k+1}$

Observe: $a^{k+1} = \underbrace{a^k \cdot a \leq b^k \cdot a}_{\text{by Ind. Hyp.}} \leq \underbrace{b^k \cdot b}_{a \leq b} = b^{k+1}$

i.e., $a^{k+1} \leq b^{k+1}$

Hence, $a^n \leq b^n$ for all natural numbers, n. ■

2. Given that $\frac{d}{dx}\left[x^0\right] = 0$ and $\frac{d}{dx}\left[x^1\right] = 1$, prove that $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$. You may use the product rule: $\frac{d}{dx}\left[f(x)\, g(x)\right] = f'(x)\, g(x) + g'(x)\, f(x)$.

Proof.

Step 1 Show true for $n = 1$.

$\frac{d}{dx}\left[x^1\right] = 1 = x^0 = x^{1-1}$ True.

Step 2 Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $\frac{d}{dx}\left[x^k\right] = kx^{k-1}$ and show that $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\, x^{(k+1)-1}$

i.e., show that $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\, x^k$

Observe:

$$\frac{d}{dx}\left[x^{k+1}\right] = \frac{d}{dx}\left[x^k \cdot x\right] = \underbrace{\frac{d}{dx}\left[x^k\right] \cdot x + \frac{d}{dx}\left[x\right] \cdot x^k}_{\text{product rule}} = \underbrace{kx^{k-1}}_{\text{Ind Hyp}} \cdot x + \underbrace{1}_{\text{given}} \cdot x^k$$

$= kx^k + x^k = (k+1)\, x^k$

i.e. $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\, x^k$

Hence, $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$ for all natural numbers n. ■

For problems 3 - 4, prove one using Mathematical Induction:

3. $(1+x)^n \geq 1+nx$ for any natural number n and any real number $x \geq -1$.

Proof.

Step #1: Show true for $n=1$

$(1+x)^1 = 1+x \geq 1+(1)\,x$ True.

Step #2: Assume true for $n=k$, and show true for $n=k+1$

i.e., Assume that $(1+x)^k \geq 1+kx$ for some natural number k, and show that

$(1+x)^{k+1} \geq 1+(k+1)\,x$

Observe:

$$(1+x)^{k+1} = \underbrace{(1+x)^k(1+x) \geq (1+kx)(1+x)}_{\text{by Induction Hypothesis}} = 1+kx+x+kx^2$$

$$= 1+(k+1)\,x + \underbrace{kx^2}_{kx^2 \geq 0} \geq 1+(k+1)\,x$$

i.e., $(1+x)^{k+1} \geq 1+(k+1)\,x$

Hence, $(1+x)^n \geq 1+nx$ for all natural numbers n and any real number $x \geq -1$ ■

Remark: Our proof hinged on two subtle points:

First, since k is a natural number (hence greater than zero) and $x^2 \geq 0$ for ALL real numbers x, it follows that $kx^2 \geq 0$.

Second, since it is given that $x \geq -1$ (or equivalently, $(1+x) \geq 0$), the direction of the inequality, $(1+x)^k \geq 1+kx$, is preserved when both sides are multiplied by $(1+x)$ during the application of the induction hypothesis.

4. Given that $|x_1 + x_2| \leq |x_1| + |x_2|$ (the Triangle Inequality); Prove by induction that: $|x_1 + x_2 + x_3 + \ldots + x_n| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_n|$ (the General Triangle Inequality).

Proof.

Step #1: Show that the proposition is true for $n = 1$.

$|x_1| \leq |x_1|$. True.

Step #2: Assume that the proposition is true for $n = k$, and prove that the proposition is true for $n = k + 1$.

i.e., Assume that $|x_1 + x_2 + x_3 + \ldots + x_k| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k|$ and show that $|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

Observe: $\underbrace{|(x_1 + x_2 + x_3 + \ldots + x_k) + x_{k+1}| \leq |x_1 + x_2 + x_3 + \ldots + x_k| + |x_{k+1}|}_{\text{from Triangle Inequality}}$

$\leq \underbrace{|x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|}_{\text{by Ind. Hyp.}}$.

i.e., $|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

Hence, $|x_1 + x_2 + x_3 + \ldots + x_n| \leq |x_1| + |x_2| + |x_3| + \ldots + |x_n|$ for all natural numbers, n. ■

For problems 5- 6 prove one using Mathematical Induction.

5. $2 + 4 + 6 + \ldots + 2n = n^2 + n$
   i.e. $\sum_{i=1}^{n} 2i = n^2 + n$

   Proof.

   Step #1: Show true for $n = 1$

   $\sum_{i=1}^{1} 2i = 2(1) = 2 = (1)^2 + (1)$ True.

   Step #2: Assume true for $n = k$, and show true for $n = k + 1$

   i.e., Assume that $\sum_{i=1}^{k} 2i = k^2 + k$ for some natural number k, and show that $\sum_{i=1}^{k+1} 2i = (k+1)^2 + (k+1)$

   Observe:

$$\sum_{i=1}^{k+1} 2i = \underbrace{\sum_{i=1}^{k} 2i + 2(k+1) = \left(k^2 + k\right) + 2(k+1)}_{\text{by Induction Hypothesis}}$$

$$= (k^2 + k) + 2k + 2 = k^2 + k + (k+1) + (k+1) = k^2 + 2k + 1 + (k+1)$$

$$= (k+1)^2 + (k+1)$$

   i.e., $\sum_{i=1}^{k+1} 2i = (k+1)^2 + (k+1)$

   Hence, $\sum_{i=1}^{n} 2i = n^2 + n$ for all natural numbers, n. ■

6. $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

i.e. $\sum_{j=1}^{n}\frac{1}{(2j-1)(2j+1)}=\frac{n}{2n+1}$

Proof.

Step #1: Show true for $n=1$

$\sum_{i=1}^{1}\frac{1}{(2i-1)(2i+1)}=\frac{1}{(2(1)-1)(2(1)+1)}=\frac{1}{3}=\frac{(1)}{2(1)+1}$ True.

Step #2: Assume true for $n=k$, and show true for $n=k+1$

i.e., Assume that $\sum_{i=1}^{k}\frac{1}{(2i-1)(2i+1)}=\frac{k}{2k+1}$ for some natural number k, and show that $\sum_{i=1}^{k+1}\frac{1}{(2i-1)(2i+1)}=\frac{k+1}{2(k+1)+1}$

i.e., $\sum_{i=1}^{k+1}\frac{1}{(2i-1)(2i+1)}=\frac{k+1}{2k+3}$

Observe:

$$\sum_{i=1}^{k+1}\frac{1}{(2i-1)(2i+1)}=\sum_{i=1}^{k}\frac{1}{(2i-1)(2i+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}\text{ (by Induction Hypothesis)}$$

$$=\frac{k}{2k+1}\cdot\frac{2k+3}{2k+3}+\frac{1}{(2k+1)(2k+3)}=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}=\frac{(k+1)}{(2k+3)}$$

i.e., $\sum_{i=1}^{k+1}\frac{1}{(2i-1)(2i+1)}=\frac{k+1}{2k+3}$

Hence, $\sum_{i=1}^{n}\frac{1}{(2i-1)(2i+1)}=\frac{n}{2n+1}$ for all natural numbers, n. ■

For problems 7- 9 prove one using Mathematical Induction.

7. $2+6+10+\ldots+4n-2=2n^2$

i.e., $\sum_{i=1}^{n}(4i-2)=2n^2$

Proof.

Step #1: Show true for $n=1$

$\sum_{i=1}^{1}(4i-2)=(4(1)-2)=2=2(1)^2$ True.

Step #2: Assume true for $n=k$, and show true for $n=k+1$

i.e., Assume that $\sum_{i=1}^{k}(4i-2)=2k^2$ for some natural number k, and show that $\sum_{i=1}^{k+1}(4i-2)=2(k+1)^2$

Observe:

$$\sum_{i=1}^{k+1}(4i-2)=\underbrace{\sum_{i=1}^{k}(4i-2)+[4(k+1)-2]=2k^2+[4(k+1)-2]}_{\text{by Induction Hypothesis}}$$

$$=2k^2+4k+4-2$$

$$=2k^2+4k+2$$

$$=2(k^2+2k+1)$$

$$=2(k+1)^2$$

i.e., $\sum_{i=1}^{k+1}(4i-2)=2(k+1)^2$

Hence, $\sum_{i=1}^{n}(4i-2)=2n^2$ for all natural numbers, n. ■

8. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

i.e. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

i.e. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Proof.

Step #1: Show true for $n = 1$

$\sum_{i=1}^{1} i^3 = (1)^3 = 1 = \frac{(1)^2((1)+1)^2}{4}$ True.

Step #2: Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$ for some natural number k, and show that

$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$

i.e., $\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$

Observe:

$$\sum_{i=1}^{k+1} i^3 = \underbrace{\sum_{i=1}^{k} i^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3}_{\text{by Induction Hypothesis}} = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$$= \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right] = \frac{(k+1)^2}{4}\left[k^2 + 4k + 4\right] = \frac{(k+1)^2(k+2)^2}{4}$$

i.e., $\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$

Hence, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all natural numbers, n. ■

9. $\frac{n^4}{4} < 1^3 + 2^3 + 3^3 + \ldots + n^3$ all natural numbers, n.

i.e. $\frac{n^4}{4} < \sum_{i=1}^{n} i^3$

Proof.

Step #1: Show true for $n = 1$

$\frac{1^4}{4} < 1^3 = \sum_{i=1}^{1} i^3$

Step #2: Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that: $\frac{k^4}{4} < \sum_{i=1}^{k} i^3$ for some natural number k,

and show that: $\frac{(k+1)^4}{4} < \sum_{i=1}^{k+1} i^3$

Equivalently: Assume that: $\sum_{i=1}^{k} i^3 > \frac{k^4}{4}$ for some natural number k,

and show that $\sum_{i=1}^{k+1} i^3 > \frac{(k+1)^4}{4}$

Observe:

$$\sum_{i=1}^{k+1} i^3 = \underbrace{\sum_{i=1}^{k} i^3 + (k+1)^3 > \frac{k^4}{4} + (k+1)^3}_{\text{by Induction Hypothesis}} = \frac{k^4}{4} + \frac{4(k+1)^3}{4} = \frac{k^4+4k^3+12k^2+12k+4}{4}$$

$$> \frac{k^4+4k^3+6k^2+4k+1}{4} = \frac{(k+1)^4}{4}$$

i.e., $\sum_{i=1}^{k+1} i^3 > \frac{(k+1)^4}{4}$

Hence, $\frac{n^4}{4} < \sum_{i=1}^{n} i^3$ for all natural numbers, n. ∎

10. In exercises 10.a - 10.d, let p be the statement: “He practices every night,” and let q be the statement: “he will achieve greatness.” Write each statement in symbolic form.

   (a) If he practices every night, then he will achieve greatness.

   $$\text{If } \underbrace{\text{he practices every night,}}_{p} \; \underbrace{\text{then}}_{\rightarrow} \; \underbrace{\text{he will achieve greatness.}}_{q}$$

   $$\boxed{p \rightarrow q}$$

   (b) He will practice every night, or he will not achieve greatness.

   $$\underbrace{\text{He will practice every night,}}_{p} \; \underbrace{\text{ or }}_{\vee} \; \underbrace{\text{he will not achieve greatness.}}_{\sim q}$$

   $$\boxed{p \vee (\sim q)}$$

   (c) His practicing every night is a necessary and sufficient condition for him to achieve greatness.

   $$\underbrace{\text{His practicing every night}}_{p} \; \underbrace{\text{is a necessary and sufficient condition for}}_{\leftrightarrow} \; \underbrace{\text{him to achieve greatness.}}_{q}$$

   $$\boxed{p \leftrightarrow q}$$

   (d) He will have practiced every night if he achieves greatness.

   $$\underbrace{\text{He will have practiced every night}}_{p} \; \underbrace{\text{ if }}_{\leftarrow} \; \underbrace{\text{he achieves greatness.}}_{q}$$

   $$\boxed{p \leftarrow q \qquad \text{Alternatively:} \qquad q \rightarrow p}$$

11. In exercises 11.a - 11.d, let p be the statement: "Fall came early this year," and let q be the statement: "I love the weather." Write each statement in words.

   (a) $p \wedge q$

   $\underbrace{\text{Fall came early this year}}_{p}\underbrace{\text{and}}_{\wedge}\underbrace{\text{I love the weather.}}_{q}$

   (b) $p \vee q$

   $\underbrace{\text{Fall came early this year}}_{p}\underbrace{\text{ or }}_{\vee}\underbrace{\text{I love the weather.}}_{q}$

   (c) $q \rightarrow \sim p$

   $\underbrace{\text{If I love the weather,}}_{q}\underbrace{\text{then}}_{\rightarrow}\underbrace{\text{Fall did not come early this year.}}_{\sim p}$

   (d) $\sim p \leftrightarrow \sim q$

   $\underbrace{\text{Fall will not come early this year}}_{\sim p}\underbrace{\text{if and only if}}_{\leftrightarrow}\underbrace{\text{I do not love the weather.}}_{\sim q}$

   Alternatively:
   $\underbrace{\text{Fall did not come early this year}}_{\sim p}\underbrace{\text{if and only if}}_{\leftrightarrow}\underbrace{\text{I did not love the weather.}}_{\sim q}$

12. In problems 12.a - 12.d, determine whether the given propositions are True or False:

(a) If $8 + 3 = 9$, then $7 < 10$.

$$\text{If} \underbrace{8+3=9,}_{F} \underbrace{\text{then}}_{\rightarrow} \underbrace{7<10}_{T} = T$$

True

(b) If $8 > 3$, then $8 > 5$.

$$\text{If} \underbrace{8>3}_{T} \underbrace{\text{then}}_{\rightarrow} \underbrace{8>5}_{T} = T$$

True

(c) $8 > 10$ if and only if $2 + 2 = 5$.

$$\underbrace{8>10}_{F} \underbrace{\text{if an only if}}_{\leftrightarrow} \underbrace{2+2=5}_{F} = T$$

True

(d) If $2 + 2 = 4$, then $8 > 10$.

$$\text{If} \underbrace{2+2=4,}_{T} \underbrace{\text{then}}_{\rightarrow} \underbrace{8>10}_{F} = F$$

False

13. In exercises 13.a-13.b construct a truth table for the statement given.

(a) $p \longleftrightarrow (q \wedge r)$

p	q	r	$(q \wedge r)$	$p \longleftrightarrow (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	T	T	T	F
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

(b) $(\sim p \vee q) \to r$

p	q	r	$\sim p$	$(\sim p \vee q)$	$(\sim p \vee q) \to r$
T	T	T	F	T	T
T	T	F	F	T	F
T	F	T	F	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	F
F	F	T	T	T	T
F	F	F	T	T	F