MTH 1126 - Test #4 - Version 2 - Solutions

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Show CLEARLY how you arrive at your answers.

In Exercises 1-2, Determine convergence/divergence. If the integral converges, find its value.

1. $\int_2^\infty \frac{1}{(x-1)} dx =$

$$\int_2^\infty \frac{1}{(x-1)} dx = \lim_{b\to\infty} \int_2^b \underbrace{\frac{1}{(x-1)}}_{\frac{1}{u}} \underbrace{dx}_{du} = \lim_{b\to\infty} \left[\ln(x-1)\right]_2^b$$

$$= \lim_{b\to\infty} \left[\ln(b) - \ln(2-1)\right] = \lim_{b\to\infty} \left[\ln(b) - 0\right] = \infty$$

i.e., $\int_2^\infty \frac{1}{(x-1)} dx = \infty$ (Integral **Diverges)**

2. $\int_0^3 \frac{1}{\sqrt{3-x}} dx =$

Because $\frac{1}{\sqrt{3-x}}$ is discontinuous at $x = 3$, this is an improper integral.

$$\int_0^3 \frac{1}{\sqrt{3-x}} dx = \lim_{b\to 3^-} \int_0^b \frac{1}{(3-x)^{\frac{1}{2}}} dx = \lim_{b\to 3^-} \int_0^b \underbrace{(3-x)^{-\frac{1}{2}}}_{u^{-\frac{1}{2}}} \underbrace{dx}_{-du} dx = \lim_{b\to 3^-} \left[-\frac{(3-x)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right]_0^b$$

$$= \lim_{b\to 3^-} \left[-2(3-x)^{\frac{1}{2}}\right]_0^b = \lim_{b\to 3^-} \left[-2(3-b)^{\frac{1}{2}} - \left(-2(3-0)^{\frac{1}{2}}\right)\right]$$

$$= \lim_{b\to 3^-} \left[-2(3-b)^{\frac{1}{2}} + 2(3-0)^{\frac{1}{2}}\right] = \left[2(0)^{\frac{1}{2}} + 2(3)^{\frac{1}{2}}\right] = 2\sqrt{3}$$

i.e. $\int_0^3 \frac{1}{\sqrt{3-x}} dx = 2\sqrt{3}$ (Integral **Converges)**

3. Determine convergence/divergence of the sequence whose n^{th} term is given by: $a_n = \frac{1+(-1)^n}{n}$. (i.e., Determine convergence/divergence of the sequence $\left\{\frac{1+(-1)^n}{n}\right\}_{n=1}^{\infty} = \left\{0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, \ldots\right\}$.)

   Observe: When n is odd, $a_n = \frac{1+(-1)^n}{n} = a_n = \frac{1+(-1)}{n} = 0$

   So when n is odd, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 0 = 0$

   When n is even, $a_n = \frac{1+(-1)^n}{n} = \frac{1+1}{n} = \frac{2}{n}$

   So when n is even, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2}{n} = 0$

   Thus, $\lim_{n\to\infty} a_n = 0$

   $\boxed{\lim_{n\to\infty} \frac{1+(-1)^n}{n} = 0 \text{ (i.e., The sequence } \textbf{Converges} \text{ to 0.)}}$

4. Determine convergence/divergence of the given series. (Justify your answer!) **If the series converges, determine its sum.**

   $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} =$

   If the series converges, determine its sum. In general, there are only two types of convergent series whose sums we can compute: Geometric and "Telescoping Sum."

   The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6}$ is definitely NOT Geometric.

   Maybe it can be written as a "Telescoping Sum."

   So let's see if we can express $a_n = \frac{1}{n^2+3n+2}$ as the difference of two terms.

   $\frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)} = \frac{C_1}{n+1} + \frac{C_2}{(n+2)}$

   i.e., $\frac{1}{(n+1)(n+2)} = \frac{C_1}{n+1} + \frac{C_2}{n+2}$

   $\Rightarrow \frac{1}{(n+1)(n+2)}(n+1)(n+2) = \frac{C_1}{n+1}(n+1)(n+2) + \frac{C_2}{n+2}(n+1)(n+2)$

   $\Rightarrow 1 = C_1(n+2) + C_2(n+1)$

   $\boxed{n=-2}$ $\Rightarrow 1 = C_2(-1)$

   $\boxed{\Rightarrow C_2 = -1}$

   $\boxed{n=-1}$ $\Rightarrow 1 = C_1(1)$

$$\boxed{\Rightarrow C_1 = 1}$$

Thus, $\frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$

$$\Rightarrow \sum_{n=1}^{N} \frac{1}{n^2+3n+2} = \sum_{n=1}^{N} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots$$

$$+ \left(\frac{1}{N-1} - \frac{1}{N}\right) + \left(\frac{1}{N} - \frac{1}{N+1}\right) + \left(\frac{1}{N+1} - \frac{1}{N+2}\right)$$

$$= \sum_{n=1}^{N} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{2} - \frac{1}{N+2}$$

i.e., $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} = \lim_{N\to\infty} \sum_{n=1}^{N} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{N+2}\right) = \frac{1}{2}$

$$\boxed{\text{i.e., } \sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} = \frac{1}{2}}$$

In Exercises 5-6, determine convergence/divergence of the given series. (Justify your answers!) **If the series converges, determine its sum.**

5. $1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \frac{16}{625} + \ldots$

   If the series converges, determine its sum. In general, there are only two types of convergent series whose sums we can compute: Geometric and "Telescoping Sum."

   Notice that each term after the first term is equal to $\frac{2}{5}$ times its predecessor.

   The series is geometric with ratio $r = \frac{2}{5}$

   Since $|r| < 1$, the series converges to $\frac{1^{\text{st}} \text{ term}}{1-r} = \frac{1}{1-\frac{2}{5}} = \frac{1}{\left(\frac{3}{5}\right)} = \frac{5}{3}$

   $\boxed{\text{The series } \textbf{converges} \text{ to } \frac{5}{3}}$

6. $\sum_{n=1}^{\infty} \frac{n^2+2n}{n^2+4n+3} =$

First, note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2+2n}{n^2+4n+3} = \lim_{n\to\infty} \frac{2n+2}{2n+4} = \lim_{n\to\infty} \frac{2}{2} = 1$

Since $\lim_{n\to\infty} a_n \neq 0$, the series **diverges.**

i.e., $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+5}$ **diverges** by the "n^{th} term Test."

In Exercises 7-8, determine convergence/divergence of the given series. (Justify your answers!)

7. $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{3}{2}}-1}$

We can compare $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{3}{2}}-1}$ to $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, which is a p-series with $p = \frac{3}{2}$.

Hence $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges.

Since $\underbrace{\frac{1}{n^{\frac{3}{2}}}}_{a_n} < \underbrace{\frac{1}{n^{\frac{3}{2}}-1}}_{b_n}$ and $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges, we can conclude nothing about $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{3}{2}}-1}$.

The **Direct Comparison Test** is inconclusive.

However: Observe that $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\left(\frac{1}{n^{\frac{3}{2}}}\right)}{\left(\frac{1}{n^{\frac{3}{2}}-1}\right)}\right| = \lim_{n\to\infty} \frac{n^{\frac{3}{2}}-1}{n^{\frac{3}{2}}} = 1$

Since $0 < \lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| < \infty$, Both series "do the same thing."

Since $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, is a convergent p-series (with $p = \frac{3}{2}$), $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{3}{2}}-1}$ converges also, by the **Limit**

Comparison Test.

i.e., $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{3}{2}}-1}$ **converges** by **Limit Comparison** with $\sum_{n=4}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

8. $\sum_{n=1}^{\infty} \frac{1}{n+3}$

There are a few ways to do this.

First, we can compare $\sum_{n=1}^{\infty} \frac{1}{n+3}$ with $\sum_{n=1}^{\infty} \frac{1}{n}$, which is the divergent Harmonic Series.

Since $\underbrace{\frac{1}{n+3}}_{a_n} < \underbrace{\frac{1}{n}}_{b_n}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the Direct Comparison Test doesn't apply.

Applying the Limit Comparison Test, we have: $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left|\frac{\left(\frac{1}{n+3}\right)}{\left(\frac{1}{n}\right)}\right| = \lim_{n\to\infty} \frac{n}{n+3} = 1$

Since $0 < \lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| < \infty$, Both series "do the same thing."

Since $\sum_{n=1}^{\infty} \frac{1}{n}$, is the divergent Harmonic Series, $\sum_{n=4}^{\infty} \frac{1}{n+3}$ diverges also, by the **Limit Comparison Test.**

Alternatively, $\int_1^{\infty} \frac{1}{n+3} dn = \lim_{b\to\infty} \int_1^b \underbrace{\frac{1}{n+3}}_{\frac{1}{u}} \underbrace{dn}_{du} = \lim_{b\to\infty} \left[\ln(n+3)\right]_1^b =$

$\lim_{b\to\infty} \left[\ln(b+3) - \ln(1)\right] = \infty$

$\sum_{n=4}^{\infty} \frac{1}{n+3}$ **diverges** by the **Integral Test**

> $\sum_{n=4}^{\infty} \frac{1}{n+3}$ **diverges** by the **Integral Test** and by **Limit Comparison** with $\sum_{n=4}^{\infty} \frac{1}{n}$

For exercises 9-10, choose one. (You can do the other for extra credit. (10 points))

9. Determine convergence/divergence of the given series. (Justify your answer!)

   $\sum_{n=1}^{\infty} \left(\frac{n+1}{3n+2}\right)^n$

   The n^{th} term, a_n is something **raised to the n^{th} power,** so this series is a good candidate for the n^{th} **Root Test.**

   Observe: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{3n+2}\right)^n} = \lim_{n\to\infty} \left(\frac{n+1}{3n+2}\right) = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3}$

   Since $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, the series **converges.** by the n^{th} **Root Test.**

   $\sum_{n=1}^{\infty} \left(\frac{n+1}{3n+2}\right)^n$ **converges** by the n^{th} **Root Test.**

10. Determine convergence/divergence of the given series. (Justify your answer!)

    $\sum_{n=1}^{\infty} \frac{3^n}{n!}$

    The n^{th} term a_n contains a **factorial,** so this is a good candidate for the **Ratio Test.**

    Observe: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{3^{n+1}}{(n+1)!}}{\left(\frac{3^n}{n!}\right)}\right| = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!}\frac{n!}{3^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0$

    Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, the series **converges.**

    $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ **converges** by the **Ratio Test.**

Extra Wow! (10 points)

Determine convergence/divergence of the given series. (Justify your answer!)

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{2} + \dots$

Observe: Our series fits the form: $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\underbrace{\frac{1}{\sqrt{n}}}_{a_n} = \sum_{n=1}^{\infty}(-1)^{n+1}a_n$

The terms of the series are alternately positive and negative.

Also: $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ (i.e., $\lim_{n\to\infty} a_n = 0$)

And: $\displaystyle\underbrace{\frac{1}{\sqrt{n+1}}}_{a_{n+1}} \leq \underbrace{\frac{1}{\sqrt{n}}}_{a_n}$

By the Alternating Series Test, the series converges.

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ **converges** by the **Alternating Series Test**