MTH 1126 - Test #1 - Solutions

SPRING 2006

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\int_0^1 (x^3+1)^4 x^2\,dx =$

   1. Is U-sub appropriate?

      a Is there a composite function?

      Yes! $\underbrace{\left(x^3+1\right)}_{\text{inner}}\underbrace{{}^4}_{\text{outer}}$

      Let u be the "inner" function

      $\Rightarrow u = x^3+1$

      b Is there an approxiamte function/derivative pair?

      Yes! $\underbrace{\left(x^3+1\right)}_{\text{function}} \rightarrow \underbrace{x^2}_{\text{deriv}}$

      Let u be the "function" of the function/derivative pair

      $\Rightarrow u = x^3+1$

   2. Compute du

$$\boxed{\begin{array}{lcl} & u & = & x^3+1 \\ \Rightarrow & \frac{du}{dx} & = & 3x^2 \\ \Rightarrow & du & = & 3x^2dx \\ \Rightarrow & \frac{1}{3}du & = & x^2dx \end{array}}$$

$$\boxed{\begin{array}{lll} \text{When } x=0, & u=x^3+1 & = & 1 \\ \text{When } x=1, & u=x^3+1 & = & 2 \end{array}}$$

   3. Analyze Integral in terms of u and du

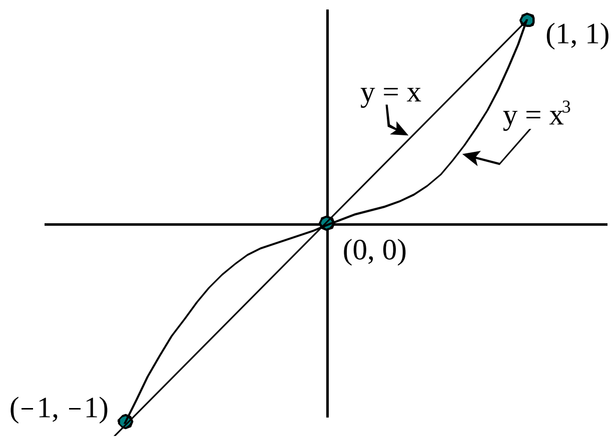
$$\int_{x=0}^{x=1} \underbrace{\left(x^3+1\right)^4}_{u^4} \underbrace{x^2\,dx}_{\frac{1}{3}du} = \int_{u=}^{u=} u^4 \tfrac{1}{3}du = \tfrac{1}{3}\int_{u=1}^{u=2} u^4 du$$

   4. Integrate (in terms of u)

$$\tfrac{1}{3}\int_{u=1}^{u=2} u^4du = \tfrac{1}{3}\left[\tfrac{u^5}{5}\right]_{u=1}^{u=2} = \tfrac{1}{3}\left[\left(\tfrac{2^5}{5}\right) - \left(\tfrac{1^5}{5}\right)\right] = \tfrac{31}{15}$$

2. Use the "$f - g$" method to compute the area bounded by the graphs of $f(x) = x^3$ and $g(x) = x$

   1. First, graph the bounded region:



   2. Observe: From $x = -1$ to $x = 0$, $x^3 \geq x$, and from From $x = 0$ to $x = 1$, $x \geq x^3$.

      Hence area from $x = -1$ to $x = 0$ is given by $\int_{-1}^{0} (x^3 - x)\, dx$

      and area from $x = 0$ to $x = 1$ is given by $\int_{0}^{1} (x - x^3)\, dx$

      Therefore, total bounded area is given by:

      (area from $x = -1$ to $x = 0$) + (area from $x = 0$ to $x = 1$)

      $= \int_{-1}^{0} (x^3 - x)\, dx + \int_{0}^{1} (x - x^3)\, dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1}$

      $= \left[\left(\frac{0^4}{4} - \frac{0^2}{2}\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right)\right] + \left[\left(\frac{1^2}{2} - \frac{1^4}{4}\right) - \left(\frac{0^2}{2} - \frac{0^4}{4}\right)\right] = \frac{1}{2}$

3. Suppose that $\int_2^8 (f(x) - g(x))\,dx = 9$; $\int_2^8 g(x)\,dx = 5$; and that $\int_4^2 f(x)\,dx = 4$. Compute $\int_4^8 f(x)\,dx$.

Since $\underbrace{\int_4^2 f(x)\,dx}_{\text{known}} + \underbrace{\int_2^8 f(x)\,dx}_{\text{unknown}} = \underbrace{\int_4^8 f(x)\,dx}_{\text{what we want}}$, our first order of business is to find $\int_2^8 f(x)\,dx$.

Observe: $\int_2^8 (f(x) - g(x))\,dx + \int_2^8 g(x)\,dx = \int_2^8 f(x)\,dx - \int_2^8 g(x)\,dx + \int_2^8 g(x)\,dx$

$= \int_2^8 f(x)\,dx$

i.e., $\int_2^8 f(x)\,dx = \underbrace{\int_2^8 (f(x) - g(x))\,dx}_{9} + \underbrace{\int_2^8 g(x)\,dx}_{5} = 9 + 5 = 14$

i.e., $\int_2^8 f(x)\,dx = 14$

Next observe: $\underbrace{\int_4^2 f(x)\,dx}_{4} + \underbrace{\int_2^8 f(x)\,dx}_{14} = \int_4^8 f(x)\,dx$

i.e., $\int_4^8 f(x)\,dx = 4 + 14 = 18$

4. Compute: $\int \tan(x^2)\sec^2(x^2)\,x\,dx =$

   1. Is U-sub appropriate?

      a Is there a composite function?

      $???\underbrace{\tan}_{\text{outer}}\underbrace{\left(x^2\right)}_{\text{inner}};\ \underbrace{\sec^2}_{\text{outer}}\underbrace{\left(x^2\right)}_{\text{inner}};\ \underbrace{(\sec(x))}_{\text{inner}}\underbrace{{}^2}_{\text{outer}}$ these are all composite functions - How do we choose???

      b Is there an approxiamte function/derivative pair?

      Yes. $\underbrace{x^2}_{\text{function}} \rightarrow \underbrace{x}_{\text{deriv}}$ but ALSO $\underbrace{\tan\left(x^2\right)}_{\text{function}} \rightarrow \underbrace{\sec^2\left(x^2\right)x}_{\text{deriv}}$

      For now, we'll try the second pair.

      Let u be the "function" of the function/derivative pair

      $\Rightarrow u = \tan(x^2)$

   2. Compute du

	u	$=$	$\tan(x^2)$
$\Rightarrow$	$\frac{du}{dx}$	$=$	$\sec^2(x^2)\cdot 2x = 2\sec^2(x^2)\,x$
$\Rightarrow$	du	$=$	$2\sec^2(x^2)\,x\,dx$
$\Rightarrow$	$\frac{1}{2}du$	$=$	$\sec^2(x^2)\,x\,dx$

   3. Analyze Integral in terms of u and du

      $\int \underbrace{\tan\left(x^2\right)}_{u}\underbrace{\sec^2\left(x^2\right)x\,dx}_{\frac{1}{2}du} = \int u\cdot\frac{1}{2}\,du = \frac{1}{2}\int u\,du$

   4. Integrate (in terms of u)

      $\frac{1}{2}\int u\,du = \frac{1}{2}\left[\frac{u^2}{2}\right] + C = \frac{1}{4}u^2 + C$

   5. Rewrite in terms of x

      $\int \tan(x^2)\sec^2(x^2)\,x\,dx = \frac{1}{4}\left(\tan(x^2)\right)^2 + C = \frac{1}{4}\tan^2(x^2) + C$

5. Find the area bounded by the graphs of $f(x) = 1 - x^2$ and $g(x) = -x - 1$. (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

   1. Graph the bounded region and find the points of intersection:

      To find the points of intersection, set the y-coordinates equal, and solve for x.

      $y = 1 - x^2 = -x - 1$

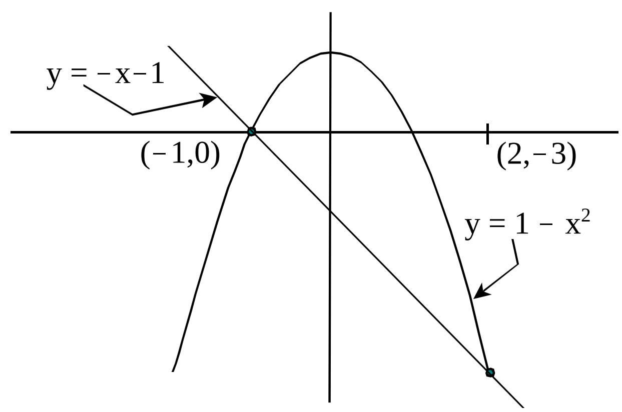
      $\Rightarrow 1 - x^2 = -x - 1$

      $\Rightarrow 2 + x - x^2 = 0$

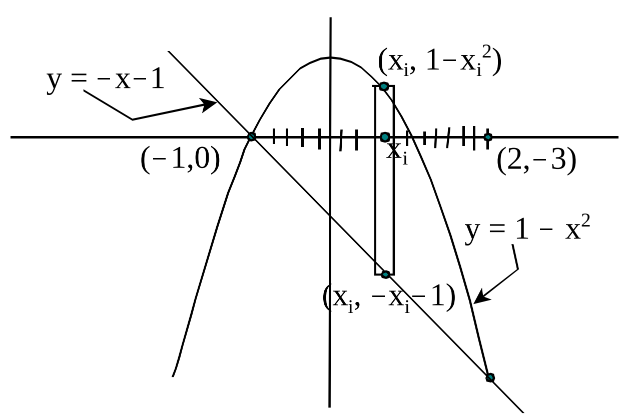
      $\Rightarrow x^2 - x - 2 = 0$

      $\Rightarrow (x + 1)(x - 2) = 0$

      $\Rightarrow x = -1, 2$ are the points of intersection



   2. Partition the interval spanned by the region and inscribe a typical rectangle of width Δx over one of the rectangles.



3. Compute the area of the i^{th} rectangle.

   Area of i^{th} rect $= \left[(1 - x_i^2) - (-x_i - 1)\right] \Delta x = (2 + x_i - x_i^2)\, \Delta x$

4. Approximate the area of the bounded region by adding up the areas of the rectangles

   Area $\approx \sum_{i=1}^{n} (2 + x_i - x_i^2)\, \Delta x$

5. Let $\Delta x \to 0$

   Area $= \lim_{\Delta x \to 0} \sum_{i=1}^{n} (2 + x_i - x_i^2)\, \Delta x = \int_{x=-1}^{x=2} (2 + x - x^2)\, dx = \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{2}$

   $= \left(2\,(2) + \frac{1}{2}\,(2)^2 - \frac{1}{3}\,(2)^3\right) - \left(2\,(-1) + \frac{1}{2}\,(-1)^2 - \frac{1}{3}\,(-1)^3\right) = \frac{9}{2}$

6. A region in the x-y plane is bounded by the graphs $y = x^2$ and $y = \sqrt{x}$. Use the Disk Method to compute the volume of the solid of revolution generated by revolving the region about the x-axis. (Partition the proper interval, build the Riemann Sum, derive the appropriate integral.)

   1. Graph the bounded region and find the points of intersection:

      To find points of intersection, set the y-coordinates equal, and solve for x.

      $y = x^2 = \sqrt{x}$

      $\Rightarrow x^2 = \sqrt{x}$

      $\Rightarrow \left(x^2\right)^2 = \left(\sqrt{x}\right)^2$

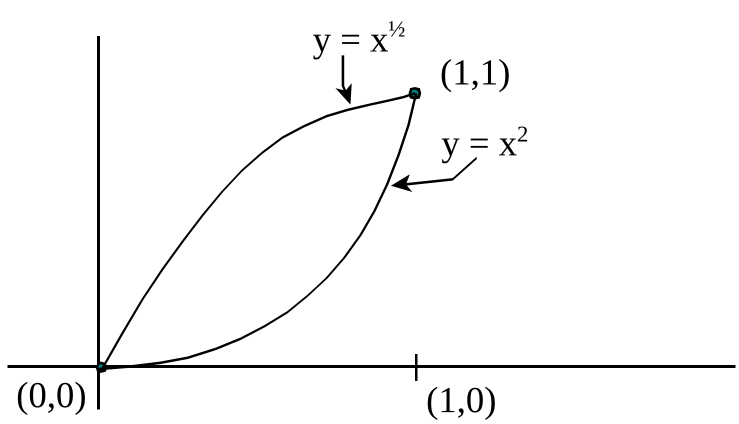
      $\Rightarrow x^4 = x$

      $\Rightarrow x^4 - x = 0$

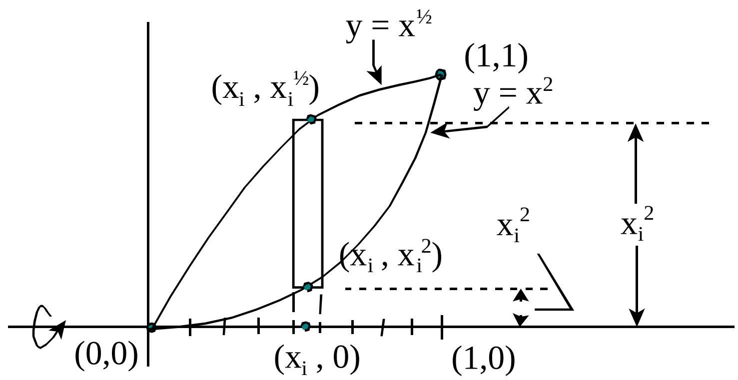
      $\Rightarrow x\,(x^3 - 1) = 0$

      $\Rightarrow x\,(x - 1)\,\underbrace{\left(x^2 + x + 1\right)}_{\text{doesn't factor}} = 0$

      $\Rightarrow x = 0, 1$ points of intersection



2. Draw a typical rectangle perpendicular to the axis of revolution and partition the interval spanned by the bounded region into sub-intervals of width Δx.



3. Compute the volume of the i^{th} washer

   Vol. i^{th} washer = (Vol. i^{th} large disk) - (Vol. i^{th} hole)

   $= \pi R_i^2 \Delta x - \pi r_i^2 \Delta x = \pi \left(x_i^{\frac{1}{2}} \right)^2 \Delta x - \pi \left(x_i^2 \right)^2 \Delta x = \pi x_i \Delta x - \pi x_i^4 \Delta x = \pi \left(x_i - x_i^4 \right) \Delta x$

4. Approximate the volume of the solid of revolution by adding the volumes of the washers

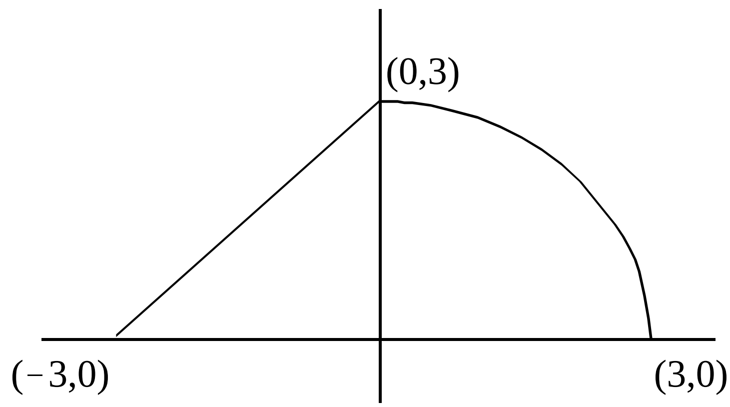
   Vol $\approx \sum_{i=1}^{n} \pi \left(x_i - x_i^4 \right) \Delta x$

5. Let $\Delta x \to 0$

   Vol $= \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi \left(x_i - x_i^4 \right) \Delta x = \int_{x=0}^{x=1} \pi \left(x - x^4 \right) dx = \pi \left[\frac{1}{2} x^2 - \frac{1}{5} x^5 \right]_0^1$

   $= \pi \left[\left(\frac{1}{2} (1)^2 - \frac{1}{5} (1)^5 \right) - \left(\frac{1}{2} (0)^2 - \frac{1}{5} (0)^5 \right) \right] = \frac{3\pi}{10}$

7. The graph of $f(x)$ is shown below. Compute $\int_{-3}^{3} f(x)\,dx$.



Observe that from $x = -3$ to $x = 3$, $f(x) \geq 0$. Therefore, $\int_{-3}^{3} f(x)\,dx$ is equal to the area bounded by the graph of $f(x)$ and the x-axis, from $x = -3$ to $x = 3$.

$\Rightarrow \int_{-3}^{3} f(x)\,dx = (\text{area of triangle}) + \left(\text{area of } \frac{1}{4} \text{ circle}\right)$

$= \frac{1}{2}(b)(h) + \frac{1}{4}\pi r^2 = \frac{1}{2}(3)(3) + \frac{1}{4}\pi(3)^2 = \frac{9}{2} + \frac{9\pi}{4}$