MTH 3318 Induction Set 1b - Solutions

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Pat Rossi Name ____________________

Instructions. Prove the following by mathematical induction.

1. Given that $|x_1 + x_2| \le |x_1| + |x_2|$ (the Triangle Inequality); Prove by induction that:

   $|x_1 + x_2 + x_3 + \ldots + x_n| \le |x_1| + |x_2| + |x_3| + \ldots + |x_n|$ for real numbers $x_1, x_2, \ldots x_n$, (the General Triangle Inequality).

   Proof.

   Step #1: Show that the proposition is true for $n = 1$.

   $|x_1| \le |x_1|$. True.

   Step #2: Assume that the proposition is true for $n = k$, and prove that the proposition is true for $n = k + 1$.

   i.e., Assume that $|x_1 + x_2 + x_3 + \ldots + x_k| \le |x_1| + |x_2| + |x_3| + \ldots + |x_k|$ and show that

   $|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \le |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

   Observe: $\underbrace{|(x_1 + x_2 + x_3 + \ldots + x_k) + x_{k+1}| \le |x_1 + x_2 + x_3 + \ldots + x_k| + |x_{k+1}|}_{\text{from Triangle Inequality}}$

   $\le \underbrace{|x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|}_{\text{by Ind. Hyp.}}$.

   i.e., $|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \le |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

   Hence, $|x_1 + x_2 + x_3 + \ldots + x_n| \le |x_1| + |x_2| + |x_3| + \ldots + |x_n|$ for all natural

   numbers, n. ■

2. $(1+x)^n \geq 1+nx$ for any natural number n and any real number $x \geq -1$.

Proof.

Step #1: Show true for $n=1$

$(1+x)^1 = 1+x \geq 1+(1)\,x$ True.

Step #2: Assume true for $n=k,$ and show true for $n=k+1$

i.e., Assume that $(1+x)^k \geq 1+kx$ for some natural number k, and show that

$(1+x)^{k+1} \geq 1+(k+1)\,x$

Observe:

$$(1+x)^{k+1} = \underbrace{(1+x)^k(1+x) \geq (1+kx)(1+x)}_{\text{by Induction Hypothesis}} = 1+kx+x+kx^2$$

$$= 1+(k+1)\,x + \underbrace{kx^2}_{kx^2\geq 0} \geq 1+(k+1)\,x$$

i.e., $(1+x)^{k+1} \geq 1+(k+1)\,x$

Hence, $(1+x)^n \geq 1+nx$ for all natural numbers n and any real number $x \geq -1$ ■

Remark: Our proof hinged on two subtle points:

First, since k is a natural number (hence greater than zero) and $x^2 \geq 0$ for ALL real numbers x, it follows that $kx^2 \geq 0$.

Second, since it is given that $x \geq -1$ (or equivalently, $(1+x) \geq 0$), the direction of the inequality, $(1+x)^k \geq 1+kx$, is preserved when both sides are multiplied by $(1+x)$ during the application of the induction hypothesis.

3. For $0 \le a \le b$; prove that $a^n \le b^n$.

Proof.

Step #1: Show true for $n = 1$.

$$a^1 = \underbrace{a \le b}_{\text{given}} = b^1$$

i.e., $a^1 \le b^1$ True.

Step #2: Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $a^k \le b^k$ for some natural number k, and show that $a^{k+1} \le b^{k+1}$

Observe: $a^{k+1} = \underbrace{a^k \cdot a \le b^k \cdot a}_{\text{by Ind. Hyp.}} \le \underbrace{b^k \cdot b}_{a \le b} = b^{k+1}$

i.e., $a^{k+1} \le b^{k+1}$

Hence, $a^n \le b^n$ for all natural numbers, n. ■

4. $n\,(n+1)$ is divisible by 2 for all natural numbers, n.

Proof.

First, note that a natural number n is divisible by 2 if there exists a natural number m such that $n = 2m$

Step #1: Show true for $n = 1$.

$1\,((1)+1) = 2 = 2 \cdot 1$

Thus, $n\,(n+1)$ is divisible by 2, for $n = 1$.

i.e., $1\,((1)+1) = 2 = 2 \cdot 1$ True.

Step #2: Assume true for $n = k$, and show true for $n = k+1$

i.e., Assume that $k\,(k+1)$ is divisible by 2, and show that

$(k+1)\,[(k+1)+1]$ is divisible by 2.

i.e., Assume that $k\,(k+1) = 2m$, and show that

$(k+1)\,(k+2)$ is divisible by 2.

Observe:

$$(k+1)\,(k+2) = (k+1)\,k + (k+1)\,2 = \underbrace{k\,(k+1) + 2\,(k+1) = 2m+2}_{\text{by Ind. Hyp.}} = 2\,(m+1)\,.$$

i.e., $(k+1)\,(k+2) = 2\,(m+1)\,.$

i.e., $(k+1)\,(k+2)$ is divisible by 2.

Hence, $n\,(n+1)$ is divisible by 2 for all natural numbers, n. ■

5. Given that $\frac{d}{dx}[x^0] = 0$ and $\frac{d}{dx}[x^1] = 1$, prove that $\frac{d}{dx}[x^n] = nx^{n-1}$. You may use the product rule: $\frac{d}{dx}[f(x)\,g(x)] = f'(x)\,g(x) + g'(x)\,f(x)$.

Proof.

Step #1: Show that $P(n)$ is true for $n = 1$.

$\frac{d}{dx}[x^1] = 1 = x^0 = x^{1-1}$ True.

Step #2: Assume that $P(n)$ is true for $n = k$, and show that $P(n)$ is true for $n = k+1$

i.e., Assume that $\frac{d}{dx}\left[x^k\right] = kx^{k-1}$ and show that $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^{(k+1)-1}$

i.e., show that $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^k$

Observe:

$$\frac{d}{dx}\left[x^{k+1}\right] = \frac{d}{dx}\left[x^k \cdot x\right] = \underbrace{\frac{d}{dx}\left[x^k\right] \cdot x + \frac{d}{dx}[x] \cdot x^k}_{\text{product rule}} = \underbrace{kx^{k-1}}_{\text{Ind Hyp}} \cdot x + \underbrace{1}_{\text{given}} \cdot x^k$$

$$= kx^k + x^k = (k+1)\,x^k$$

i.e. $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^k$

Hence, $\frac{d}{dx}[x^n] = nx^{n-1}$ for all natural numbers n. ■