Integrals and Natural Logarithms #7 - Solutions

Fall 2019

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1. Compute: $\int \left(2x^5 - 6x^3 + 4x + 8\right) dx =$

$\int \left(2x^5 - 6x^3 + 4x + 8\right) dx = 2\left[\frac{x^6}{6}\right] - 6\left[\frac{x^4}{4}\right] + 4\left[\frac{x^2}{2}\right] + 8x + C$

$= \frac{1}{3}x^6 - \frac{3}{2}x^4 + 2x^2 + 8x + C$

i.e., $\int \left(2x^5 - 6x^3 + 4x + 8\right) dx = \frac{1}{3}x^6 - \frac{3}{2}x^4 + 2x^2 + 8x + C$
Don't forget the "+C"

2. Compute: $\int \left(5\cos(x) - 7\sec^2(x)\right) dx =$

$\int \left(5\cos(x) - 7\sec^2(x)\right) dx = 5[\sin(x)] - 7[\tan(x)] + C$

i.e., $\int \left(5\cos(x) - 7\sec^2(x)\right) dx = 5\sin(x) - 7\tan(x) + C$
Don't forget the "+C"

3. Compute: $\int_{x=0}^{x=2} \left(8x^3 + 9x^2 + 2x\right) dx =$

$$\int_{x=0}^{x=2} \underbrace{\left(8x^3 + 9x^2 + 2x\right)}_{f(x)} dx = \underbrace{\left[8\left(\frac{x^4}{4}\right) + 9\left(\frac{x^3}{3}\right) + 2x\right]_{x=0}^{x=2}}_{F(x)} = \underbrace{\left[2x^4 + 3x^3 + 2x\right]_{x=0}^{x=2}}_{F(x)}$$

$$= \underbrace{\left[2(2)^4 + 3(2)^3 + 2(2)\right]}_{F(2)} - \underbrace{\left[2(0)^4 + 3(0)^3 + 2(0)\right]}_{F(0)} = 60$$

i.e., $\int_{x=0}^{x=2} \left(8x^3 + 9x^2 + 2x\right) dx = 60$

4. Compute: $\int \left(x^3+x^2\right)^4 \left(21x^2+14x\right) dx$

   1. Is u-sub appropriate?

      a. Is there a composite function?

      Yes! $\left(x^3+x^2\right)^4$ (A function raised to a power is always a composite function!)

      Let $u =$ the "inner" of the composite function

      $\Rightarrow u = \left(x^3+x^2\right)$

      b. Is there an (approximate) function/derivative pair?

      Yes! $\underbrace{\left(x^3+x^2\right)}_{\text{function}} - - - - - \rightarrow \underbrace{\left(21x^2+14x\right)}_{\text{deriv}}$

      Let $u =$ the "function" of the function/deriv pair

      $\Rightarrow u = \left(x^3+x^2\right)$

      c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

      (i.e., do criteria **a** and **b** suggest the same choice of u ?)

      Yes! $\Rightarrow$ u-substitution is appropriate

   2. Compute du

$$\begin{aligned} u &= x^3+x^2 \\ \Rightarrow \tfrac{du}{dx} &= 3x^2+2x \\ \Rightarrow du &= \left(3x^2+2x\right) dx \\ \Rightarrow 7du &= \left(21x^2+14x\right) dx \end{aligned}$$

   3. Analyze in terms of u and du

$$\int \underbrace{\left(x^3+x^2\right)^4}_{u^4} \underbrace{\left(21x^2+14x\right) dx}_{7du} = \int u^4 \cdot 7du = 7\int u^4 du$$

   4. Integrate (in terms of u).

$$7\int u^4 du = 7\left[\tfrac{u^5}{5}\right] + C = \tfrac{7}{5}u^5 + C$$

   5. Re-express in terms of the original variable, x.

$$\int \left(x^3+x^2\right)^4 \left(21x^2+14x\right) dx = \underbrace{\frac{7}{5}\left(x^3+x^2\right)^5 + C}_{\frac{7}{5}u^5+C}$$

$$\text{i.e., } \int \left(x^3+x^2\right)^4 \left(21x^2+14x\right) dx = \tfrac{7}{5}\left(x^3+x^2\right)^5 + C$$

5. Compute: $\int \sin(\sec(x)) \sec(x) \tan(x)\ dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\sin(\sec(x))$

   $$\underset{\text{outer}}{\nearrow} \qquad \underset{\text{inner}}{\nwarrow}$$

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \sec(x)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\sec(x)}_{\text{function}} - - - - \rightarrow \underbrace{\sec(x)\tan(x)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \sec(x)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\boxed{\begin{array}{rcl} u & = & \sec(x) \\ \Rightarrow \frac{du}{dx} & = & \sec(x)\tan(x) \\ \Rightarrow du & = & \sec(x)\tan(x)\ dx \end{array}}$$

3. Analyze in terms of u and du

$$\int \underbrace{\sin(\sec(x))}_{\sin(u)} \underbrace{\sec(x)\tan(x)\ dx}_{du} = \int \sin(u)\ du$$

4. Integrate (in terms of u).

$$\int \sin(u)\ du = -\cos(u) + C$$

5. Re-express in terms of the original variable, x.

$$\int \sin(\sec(x)) \sec(x) \tan(x)\ dx = \underbrace{-\cos(\sec(x)) + C}_{-\cos(u)+C}$$

$$\boxed{\text{i.e., } \int \sin(\sec(x)) \sec(x) \tan(x)\ dx = -\cos(\sec(x)) + C}$$

6. Compute: $\int \frac{6x^2+3x+3}{4x^3+3x^2+6x} dx =$

$$\int \frac{6x^2+3x+3}{4x^3+3x^2+6x} dx \underbrace{=}_{\text{re-write}} \int \frac{1}{4x^3+3x^2+6x} \left(6x^2+3x+3\right) dx$$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{4x^3+3x^2+6x}$ is the same as $\left(4x^3+3x^2+6x\right)^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = 4x^3 + 3x^2 + 6x$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(4x^3+3x^2+6x\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(6x^2+3x+3\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = 4x^3 + 3x^2 + 6x$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 4x^3 + 3x^2 + 6x \\ \Rightarrow \frac{du}{dx} &= 12x^2 + 6x + 6 \\ \Rightarrow du &= \left(12x^2 + 6x + 6\right) dx \\ \Rightarrow \frac{1}{2} du &= \left(6x^2 + 3x + 3\right) dx \end{aligned}$$

3. Analyze in terms of u and du

$$\underbrace{\int \frac{1}{4x^3+3x^2+6x}}_{\frac{1}{u}} \underbrace{\left(6x^2+3x+3\right) dx}_{\frac{1}{2} du} = \int \frac{1}{u} \frac{1}{2} du = \frac{1}{2} \int \frac{1}{u} du$$

4. Integrate (in terms of u).

$$\frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C$$

5. Re-express in terms of the original variable, x.

$$\int \frac{6x^2+3x+3}{(4x^3+3x^2+6x)} dx = \underbrace{\frac{1}{2} \ln\left|4x^3+3x^2+6x\right| + C}_{\frac{1}{2} \ln|u| + C}$$

$$\text{i.e., } \int \frac{6x^2+3x+3}{(4x^3+3x^2+6x)}dx = \frac{1}{2}\ln\left|4x^3+3x^2+6x\right|+C$$

7. Compute: $\frac{d}{dx}\left[\ln\left(\sin(x)+\cos(x)\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(\sin(x)+\cos(x)\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{\sin(x)+\cos(x)}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(\cos(x)-\sin(x)\right)}_{g'(x)} = \frac{\cos(x)-\sin(x)}{\sin(x)+\cos(x)}$$

i.e., $\frac{d}{dx}\left[\ln\left(\sin(x)+\cos(x)\right)\right] = \frac{\cos(x)-\sin(x)}{\sin(x)+\cos(x)}$

8. Compute: $\frac{d}{dx}\left[\ln\left(8x^2-7x+5\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(8x^2-7x+5\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{8x^2-7x+5}}_{\frac{1}{g(x)}} \cdot \underbrace{(16x-7)}_{g'(x)} = \frac{16x-7}{8x^2-7x+5}$$

i.e., $\frac{d}{dx}\left[\ln\left(8x^2-7x+5\right)\right] = \frac{16x-7}{8x^2-7x+5}$

9. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\sin(x)\cos(x)}\right)\right] = \frac{d}{dx}\left[\ln\left(\left(\sin(x)\cos(x)\right)^{\frac{1}{2}}\right)\right]$

Remark: We can compute this derivative directly, in it's current form, but it would be much easier to use the *algebraic properties* of natural logarithms to simplify the expression first.

$$\frac{d}{dx}\left[\ln\left(\left(\sin(x)\cos(x)\right)^{\frac{1}{2}}\right)\right] = \underbrace{\frac{d}{dx}\left[\frac{1}{2}\ln\left(\sin(x)\cos(x)\right)\right]}_{\ln(a^n)=n\ln(a)} = \underbrace{\frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(\sin(x)\right)+\ln\left(\cos(x)\right)\right)\right]}_{\ln(ab)=\ln(a)+\ln(b)}$$

NOW we're ready to compute the derivative!

$$\frac{d}{dx}\left[\ln\left(\left(\sin(x)\cos(x)\right)^{\frac{1}{2}}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\left[\ln\left(\sin(x)\right)+\ln\left(\cos(x)\right)\right]\right] = \frac{1}{2}\frac{d}{dx}\left[\ln\left(\sin(x)\right)+\ln\left(\cos(x)\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\sin(x)}\cdot\cos(x) + \frac{1}{\cos(x)}\left(-\sin(x)\right)\right] = \frac{1}{2}\left(\cot(x)-\tan(x)\right)$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\sin(x)\cos(x)}\right)\right] = \frac{1}{2}\left(\cot(x)-\tan(x)\right)$

10. Compute: $\int_{x=0}^{x=3} \sqrt{x+1}\,dx = \int_{x=0}^{x=3} (x+1)^{\frac{1}{2}}\ dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

      Yes! $(x+1)^{\frac{1}{2}}$ (A function raised to a power is *always* a composite function!)

      Let $u =$ the "inner" of the composite function

      $\Rightarrow u = (x+1)$

   b. Is there an (approximate) function/derivative pair?

      ?????

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

      (i.e., do criteria **a** and **b** suggest the same choice of u ?)

      No – we don't see an (approximate) function/derivative pair.

      Nevertheless, we will proceed, based on the recommendation of Criterion a. (Note: Since Criterion b was not satisfied, u-substitution may not work.)

2. Compute du

$$\begin{array}{rcl} u & = & x+1 \\ \Rightarrow \frac{du}{dx} & = & 1 \\ \Rightarrow du & = & 1\,dx \\ \text{i.e., } du & = & dx \end{array}$$

   When $x = 0$, $u = x + 1 = (0) + 1 = 1$
   When $x = 3$, $u = x + 1 = (3) + 1 = 4$

3. Analyze in terms of u and du

$$\int_{x=0}^{x=3} \underbrace{(x+1)^{\frac{1}{2}}}_{u^{\frac{1}{2}}} \underbrace{dx}_{du} = \int_{u=1}^{u=4} u^{\frac{1}{2}}\,du$$

   Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$\int_{u=1}^{u=4} u^{\frac{1}{2}}\,du = \left[\frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_{u=1}^{u=4} = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=1}^{u=4} = \underbrace{\frac{2}{3}(4)^{\frac{3}{2}}}_{F(4)} - \underbrace{\frac{2}{3}(1)^{\frac{3}{2}}}_{F(1)} = \frac{2}{3}(8) - \frac{2}{3}(1) = \frac{14}{3}$$

$$\boxed{\text{i.e., } \int_{x=0}^{x=3} \sqrt{x+1}\,dx = \frac{14}{3}}$$

Remark: It turns out, that we really DID have an (approximate) function/derivative pair.

$\int_{x=0}^{x=3} (x+1)^{\frac{1}{2}}\ dx =$ is the same as $\int_{x=0}^{x=3} (x+1)^{\frac{1}{2}} \cdot 1\,dx$

Our (approximate) function/derivative pair is: $\underbrace{(x+1)}_{\text{function}} - - - - \rightarrow \underbrace{1}_{\text{deriv}}$