MTH 1125 Test #1 - Solutions

Summer 2009

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\lim_{x\to 2}\frac{x^3-3x-3}{x^2+x+2}=$

   1. Try Plugging in:

      $\lim_{x\to 2}\frac{x^3-3x-3}{x^2+x+2}=\frac{(2)^3-3(2)-3}{(2)^2+(2)+2}=-\frac{1}{8}$

      i.e., $\lim_{x\to 2}\frac{x^3-3x-3}{x^2+x+2}=-\frac{1}{8}$

2. Compute: $\lim_{x\to 2}\frac{x^2-4x+4}{x^2+2x-8}=$

   1. Try Plugging in:

      $\lim_{x\to 2}\frac{x^2-4x+4}{x^2+2x-8}=\frac{(2)^2-4(2)+4}{(2)^2+2(2)-8}=\frac{0}{0}$ No Good - Zero Divide!

   2. Try Factoring and Cancelling:

      $\lim_{x\to 2}\frac{x^2-4x+4}{x^2+2x-8}=\lim_{x\to 2}\frac{(x-2)^2}{(x+4)(x-2)}=\lim_{x\to 2}\frac{(x-2)}{(x+4)}=\frac{(2)-2}{(2)+4}=\frac{0}{4}=0$

      i.e., $\lim_{x\to 2}\frac{x^2-4x+4}{x^2+2x-8}=0$

3. Compute: $\lim_{x\to -\infty}\frac{5x^4-5x^3+8x^2+x}{4x^3+x^2}=$

   $\lim_{x\to -\infty}\frac{5x^4-5x^3+8x^2+x}{4x^3+x^2}=\lim_{x\to -\infty}\frac{5x^4}{4x^3}=\lim_{x\to -\infty}\frac{5x}{4}=-\infty$

   i.e., $\lim_{x\to -\infty}\frac{5x^4-5x^3+8x^2+x}{4x^3+x^2}=-\infty$

4. Compute: $\lim_{x\to 3}\frac{x+5}{x^2-x-6}=$

   1. Try Plugging in:

      $\lim_{x\to 3}\frac{x+5}{x^2-x-6}=\frac{(3)+5}{(3)^2-(3)-6}=\frac{8}{0}$ No Good - Zero Divide!

2. Try Factoring and Cancelling:

   No Good!. "Factoring and Cancelling" only works when Step #1 yields $\frac{0}{0}$.

3. Analyze the one-sided limits:

   $\lim_{x\to 3^-} \frac{x+5}{x^2-x-6} = \lim_{x\to 3^-} \frac{x+5}{(x+2)(x-3)} = \frac{8}{(5)(-\varepsilon)} = \frac{\left(\frac{8}{5}\right)}{(-\varepsilon)} = -\infty$

$$\begin{array}{ll} & x \to 3^- \\ \Rightarrow & x < 3 \\ \Rightarrow & x - 3 < 0 \end{array}$$

   $\lim_{x\to 3^+} \frac{x+5}{x^2-x-6} = \lim_{x\to 3^+} \frac{x+5}{(x+2)(x-3)} = \frac{8}{(5)(+\varepsilon)} = \frac{\left(\frac{8}{5}\right)}{(+\varepsilon)} = +\infty$

$$\begin{array}{ll} & x \to 3^+ \\ \Rightarrow & x > 3 \\ \Rightarrow & x - 3 > 0 \end{array}$$

Since the one-sided limits are not equal, $\lim_{x\to 3} \frac{x+5}{x^2-x-6}$ **Does Not Exist!**

5. $f(x) = 7x^4 + 5x^3 + 3x^2 + 4x + 5;$ Compute: $f'(x)$.

   $f'(x) = 7\left(4x^3\right) + 5\left(3x^2\right) + 3\left(2x^1\right) + 4(1) + 0 = 28x^3 + 15x^2 + 6x + 4$

   i.e., $f'(x) = 28x^3 + 15x^2 + 6x + 4$

6. $\frac{d}{dx}\left[2\sin(x) - 6\cos(x)\right] =$

   $\frac{d}{dx}\left[2\sin(x) - 6\cos(x)\right] = 2(\cos(x)) - 6(-\sin(x)) = 2\cos(x) + 6\sin(x)$

   i.e., $\frac{d}{dx}\left[2\sin(x) - 6\cos(x)\right] = 2\cos(x) + 6\sin(x)$

7. Find the asymptotes and graph: $f(x) = \frac{4x^2-1}{x^2-x-6}$

Verticals

1. Find x-values that cause division by zero.

$\Rightarrow x^2 - x - 6 =$

$\Rightarrow (x+2)(x-3) = 0$

$\Rightarrow x = -2$ and $x = 3$ are possible vertical asymptotes.

2. Compute the one-sided limits.

$\lim_{x\to-2^-} \frac{4x^2-1}{x^2-x-6} = \lim_{x\to-2^-} \frac{4x^2-1}{(x+2)(x-3)} = \frac{15}{(-\varepsilon)(-5)} = \frac{15}{(\varepsilon)(5)} = \frac{3}{\varepsilon} = +\infty$

$$\begin{array}{ll} & x \to -2^- \\ \Rightarrow & x < -2 \\ \Rightarrow & x+2 < 0 \end{array}$$

$\lim_{x\to-2^+} \frac{4x^2-1}{x^2-x-6} = \lim_{x\to-2^+} \frac{4x^2-1}{(x+2)(x-3)} = \frac{15}{(\varepsilon)(-5)} = -\frac{3}{\varepsilon} = -\infty$

$$\begin{array}{ll} & x \to -2^+ \\ \Rightarrow & x > -2 \\ \Rightarrow & x+2 > 0 \end{array}$$

Since the one-sided limits are infinite, $x = -2$ *is a vertical asymptote.*

$\lim_{x\to3^-} \frac{4x^2-1}{x^2-x-6} = \lim_{x\to3^-} \frac{4x^2-1}{(x+2)(x-3)} = \frac{35}{(5)(-\varepsilon)} = \frac{7}{(-\varepsilon)} = -\infty$

$$\begin{array}{ll} & x \to 3^- \\ \Rightarrow & x < 3 \\ \Rightarrow & x-3 < 0 \end{array}$$

$\lim_{x\to3^+} \frac{4x^2-1}{x^2-x-6} = \lim_{x\to3^+} \frac{4x^2-1}{(x+2)(x-3)} = \frac{35}{(5)(\varepsilon)} = \frac{7}{(\varepsilon)} = +\infty$

$$\begin{array}{ll} & x \to 3^+ \\ \Rightarrow & x > 3 \\ \Rightarrow & x-3 > 0 \end{array}$$

Since the one-sided limits are infinite, $x = 3$ *is a vertical asymptote.*

Horizontals

Compute the limits as $x \to -\infty$ and as $x \to +\infty$

$\lim_{x\to-\infty} \frac{4x^2-1}{x^2-x-6} = \lim_{x\to-\infty} \frac{4x^2}{x^2} = \lim_{x\to-\infty} 4 = 4$

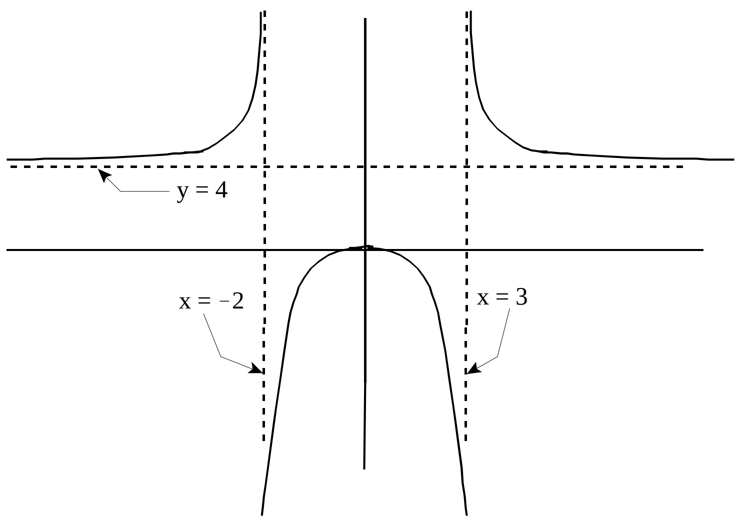
$\lim_{x\to+\infty} \frac{4x^2-1}{x^2-x-6} = \lim_{x\to+\infty} \frac{4x^2}{x^2} = \lim_{x\to+\infty} 4 = 4$

Since the limits are finite and constant, $y = 4$ is a horizontal asymptote.

Summary:

$\lim_{x\to-2^-} \frac{4x^2-1}{x^2-x-6} = +\infty$	
$\lim_{x\to-2^+} \frac{4x^2-1}{x^2-x-6} = -\infty$	$\lim_{x\to-\infty} \frac{4x^2-1}{x^2-x-6} = 4$
$\lim_{x\to3^-} \frac{4x^2-1}{x^2-x-6} = -\infty$	$\lim_{x\to+\infty} \frac{4x^2-1}{x^2-x-6} = 4$
$\lim_{x\to3^+} \frac{4x^2-1}{x^2-x-6} = +\infty$	

Graph $f(x) = \frac{4x^2-1}{x^2-x-6}$



8.

$x =$	$f(x) =$
1.5	-15.1
1.9	-227.8
1.99	-1212.3
1.999	-21156.3
1.9999	-834561.9

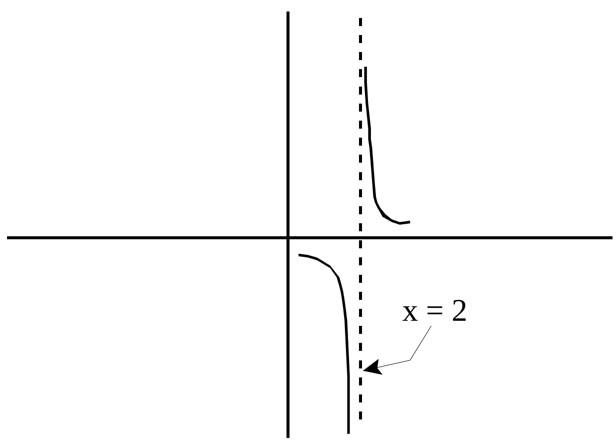
$x =$	$f(x) =$
2.5	15.1
2.1	227.8
2.01	1212.3
2.001	21156.3
2.0001	834561.9

Based on the information in the table above, do the following:

(a) $\lim_{x\to 2^-} f(x) = -\infty$

(b) $\lim_{x\to 2^+} f(x) = \infty$

(c) Graph $f(x)$



9. Determine whether or not $f(x)$ is continuous at the point $x = 3$. (Justify your answer.)

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{for } x \neq 3 \\ 9 & \text{for } x = 3 \end{cases}$$

If $f(x)$ is continuous at the point $x = 3$, then $\lim_{x\to 3} f(x) = f(3)$.

To see if this is true, we'll compute $\lim_{x\to 3} f(x)$.

Since the definiton of $f(x)$ changes at $x = 3$, we must compute the one-sided limits in order to determine whether the limit exists.

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2-9}{x-3} = \lim_{x\to 3^-} \frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3^-} (x+3) = 6$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{x^2-9}{x-3} = \lim_{x\to 3^+} \frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3^+} (x+3) = 6$

Since the one-sided limits are equal, $\lim_{x\to 3} f(x)$ exists and $\lim_{x\to 3} f(x) = 6$

However, $6 = \lim_{x\to 3} f(x) \neq f(3) = 9$

Hence, $f(x)$ is NOT continuous at $x = 3$

10. $f(x) = 3x^2 + 5x - 6$; compute $f'(x)$ **using the definition of derivative.** (i.e., compute $f'(x)$ using the "limiting process.")

$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\left[3(x+\Delta x)^2+5(x+\Delta x)-6\right]-\left(3x^2+5x-6\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\left[3\left(x^2+2x\Delta x+\Delta x^2\right)+5(x+\Delta x)-6\right]-\left(3x^2+5x-6\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\left[\left(3x^2+6x\Delta x+3\Delta x^2\right)+(5x+5\Delta x)-6\right]-\left(3x^2+5x-6\right)}{\Delta x} = \lim_{\Delta x\to 0} \frac{6x\Delta x+3\Delta x^2+5\Delta x}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\Delta x(6x+3\Delta x+5)}{\Delta x} = \lim_{\Delta x\to 0} (6x+3\Delta x+5) = 6x+3(0)+5 = 6x+5$

$\boxed{\text{i.e., } f'(x) = 6x+5}$

11. Compute: $\lim_{x\to 1} \frac{\sqrt{8+x}-3}{x-1} =$

1. Try Plugging in:

   $\lim_{x\to 1} \frac{\sqrt{8+x}-3}{x-1} = \frac{\sqrt{8+(1)}-3}{(1)-1} = \frac{0}{0}$ No Good - Zero Divide!

2. Try Factoring and Cancelling:

   $\lim_{x\to 1} \frac{\sqrt{8+x}-3}{x-1} = \lim_{x\to 1} \frac{\sqrt{8+x}-3}{x-1} \cdot \frac{\sqrt{8+x}+3}{\sqrt{8+x}+3} = \lim_{x\to 1} \frac{\left(\sqrt{8+x}\right)^2-(3)^2}{(x-1)\left[\sqrt{8+x}+3\right]}$

   $= \lim_{x\to 1} \frac{8+x-9}{(x-1)\left[\sqrt{8+x}+3\right]} = \lim_{x\to 1} \frac{(x-1)}{(x-1)\left[\sqrt{8+x}+3\right]} = \lim_{x\to 1} \frac{1}{\sqrt{8+x}+3}$

   $= \frac{1}{\sqrt{8+1}+3} = \frac{1}{3+3} = \frac{1}{6}$

   $\boxed{\text{i.e., } \lim_{x\to 1} \frac{\sqrt{8+x}-3}{x-1} = \frac{1}{6}}$

EXTRA - WOW! (7 pts) Compute: $\lim_{x\to\infty} \frac{\sin(x)}{x} =$ (Justify your answer completely)

Observe: $-1 \leq \sin(x) \leq 1$

$\Rightarrow -\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$

$\Rightarrow \underbrace{\lim_{x\to\infty} -\frac{1}{x}}_{=0} \leq \lim_{x\to\infty} \frac{\sin(x)}{x} \leq \underbrace{\lim_{x\to\infty} \frac{1}{x}}_{=0}$

i.e., $0 \leq \lim_{x\to\infty} \frac{\sin(x)}{x} \leq 0$

$\Rightarrow \lim_{x\to\infty} \frac{\sin(x)}{x} = 0$