MTH 1125 - Test 2 - Solutions

SUMMER 2023

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\frac{d}{dx}\left[6x^6+9x^4-12x^3+18x^2+8x+4\sqrt{x}+2\right]=$

   Rewrite: $\frac{d}{dx}\left[6x^6+9x^4-12x^3+18x^2+8x+4x^{\frac{1}{2}}+2\right]$

   $$=6\left(6x^5\right)+9\left(4x^3\right)-12\left(3x^2\right)+18\left(2x\right)+8+4\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right)+0$$

   $$=36x^5+36x^3-36x^2+36x+8+2x^{-\frac{1}{2}}$$

   i.e., $\frac{d}{dx}\left[6x^6+9x^4-12x^3+18x^2+8x+4\sqrt{x}+2\right]=36x^5+36x^3-36x^2+36x+8+2x^{-\frac{1}{2}}$

2. Compute: $\frac{d}{dx}\left[\left(6x^4+8x^3+5\right)\left(3x^2+6x+2\right)\right]=$

$$\frac{d}{dx}\left[\underbrace{\left(6x^4+8x^3+5\right)}_{1^{st}}\underbrace{\left(3x^2+6x+2\right)}_{2^{nd}}\right]=\underbrace{\left(24x^3+24x^2\right)}_{1^{\text{st}}\text{ prime}}\cdot\underbrace{\left(3x^2+6x+2\right)}_{2^{nd}}+\underbrace{\left(6x+6\right)}_{2^{\text{nd}}\text{ prime}}\cdot\underbrace{\left(6x^4+8x^3+5\right)}_{1^{st}}$$

   $\frac{d}{dx}\left[\left(6x^4+8x^3+5\right)\left(3x^2+6x+2\right)\right]=\left(24x^3+24x^2\right)\left(3x^2+6x+2\right)+\left(6x+6\right)\left(6x^4+8x^3+5\right)$

3. Compute: $\frac{d}{dx}\left[\frac{10x^3+15x^2+10x+2}{2x^2+8x}\right]=$

$$\frac{d}{dx}\left[\frac{\overbrace{10x^3+15x^2+10x+2}^{\text{top}}}{\underbrace{2x^2+8x}_{\text{Bottom}}}\right]=\frac{\overbrace{\left(30x^2+30x+10\right)}^{\text{top prime}}\cdot\overbrace{\left(2x^2+8x\right)}^{\text{bottom}}-\overbrace{\left(4x+8\right)}^{\text{bottom prime}}\cdot\overbrace{\left(10x^3+15x^2+10x+2\right)}^{\text{top}}}{\underbrace{\left(2x^2+8x\right)^2}_{\text{bottom squared}}}$$

   i.e., $\frac{d}{dx}\left[\frac{10x^3+15x^2+10x+2}{2x^2+8x}\right]=\frac{\left(30x^2+30x+10\right)\left(2x^2+8x\right)-\left(4x+8\right)\left(10x^3+15x^2+10x+2\right)}{\left(2x^2+8x\right)^2}$

4. Compute: $\frac{d}{dx}\left[\left(6x^4+12x^2+7\right)^5\right] =$

$$\frac{d}{dx}\left[\left(6x^4+12x^2+7\right)^5\right] = \underbrace{5\left(6x^4+12x^2+7\right)^4}_{\substack{\text{power rule} \\ \text{as usual}}} \cdot \underbrace{\left(24x^3+24x\right)}_{\substack{\text{derivative} \\ \text{of inner}}}$$

> i.e., $\frac{d}{dx}\left[\left(6x^4+12x^2+7\right)^5\right] = 5\left(6x^4+12x^2+7\right)^4\left(24x^3+24x\right)$

5. Given that $f(x) = 4x^2 - 7x + 4$, write the *equation* of the line tangent to the graph of $f(x)$ at the point $(2, 6)$.

We need two things:

i. A **point** on the line (We have that: $(x_1, y_1) = (2, 6)$)

ii. The **slope** of the line (This is $f'(x_1)$)

$f'(x) = 8x - 7$

At the point $(x_1, y_1) = (2, 6)$, **the slope is** $f'(2) = 8(2) - 7 = 9$

We will use the Point-Slope equation of a line:

$(y - y_1) = m(x - x_1)$ (Where m is the slope and (x_1, y_1) is a known point on the line.)

Thus, the equation of the line tangent to the graph of $f(x)$ is:

$(y - 6) = 9(x - 2)$

> The equation of the line tangent is $(y - 6) = 9(x - 2)$

6. Given that $v = \sin(w)$ and that $w = x^3 + 3x$; compute $\frac{dv}{dx}$ **using the Liebniz form of the Chain Rule.** (In particular, when doing this exercise, *write explicitly the Liebniz form of the chain rule that you are going to use.)*

We know:

$\frac{dv}{dw} = \cos(w)$

$\frac{dw}{dx} = 3x^2 + 3$

We want: $\frac{dv}{dx}$

By the Liebniz form of the Chain Rule:

$$\frac{dv}{dx} = \frac{dv}{dw}\frac{dw}{dx} = \cos(w)\left(3x^2+3\right) = \underbrace{\cos\left(x^3+3x\right)\left(3x^2+3\right)}_{\substack{\text{express solely in terms of} \\ \text{independent variable } x}}$$

i.e. $\frac{dv}{dx} = \cos\left(x^3+3x\right)\left(3x^2+3\right)$

7. Compute: $\frac{d}{dx}\left[\sec(\sin(x))\right] =$

Outer:	=	sec ( )
Deriv. of outer	=	sec ( ) tan ( )

$$\frac{d}{dx}\left[\begin{array}{c} \underset{\substack{\uparrow \\ \text{outer}}}{\sec}\left(\underset{\substack{\uparrow \\ \text{inner}}}{\underbrace{\sin(x)}}\right) \end{array}\right] = \underbrace{\sec(\sin(x))\tan(\sin(x))}_{\substack{\text{Deriv of outer,} \\ \text{eval. at inner}}} \cdot \underbrace{\cos(x)}_{\substack{\text{deriv. of} \\ \text{inner}}}$$

i.e., $\frac{d}{dx}\left[\sec(\sin(x))\right] = \sec(\sin(x))\tan(\sin(x))\cos(x)$

8. Compute: $\frac{d}{dx}\left[\left(\frac{3x^4+6x^2+3}{\cos(x)+4x^2}\right)^{20}\right] =$

$$\frac{d}{dx}\left[\underbrace{\left(\frac{3x^4+6x^2+3}{\cos(x)+4x^2}\right)^{20}}_{(g(x))^n}\right] = \underbrace{20\left(\frac{3x^4+6x^2+3}{\cos(x)+4x^2}\right)^{19}}_{\substack{\text{power rule}\\\text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\frac{3x^4+6x^2+3}{\cos(x)+4x^2}\right]\right)}_{\substack{\text{deriv of}\\\text{inner Function}}}$$

$$= 20\left(\frac{3x^4+6x^2+3}{\cos(x)+4x^2}\right)^{19} \underbrace{\frac{(12x^3+12x)(\cos(x)+4x^2) - (-\sin(x)+8x)(3x^4+6x^2+3)}{(\cos(x)+4x^2)^2}}_{\substack{\text{quotient}\\\text{rule}}}$$

i.e., $\frac{d}{dx}\left[\left(\frac{3x^4+6x^2+3}{\cos(x)+4x^2}\right)^{20}\right] = 20\left(\frac{3x^4+6x^2+3}{\cos(x)+4x^2}\right)^{19} \frac{(12x^3+12x)(\cos(x)+4x^2) - (-\sin(x)+8x)(3x^4+6x^2+3)}{(\cos(x)+4x^2)^2}$

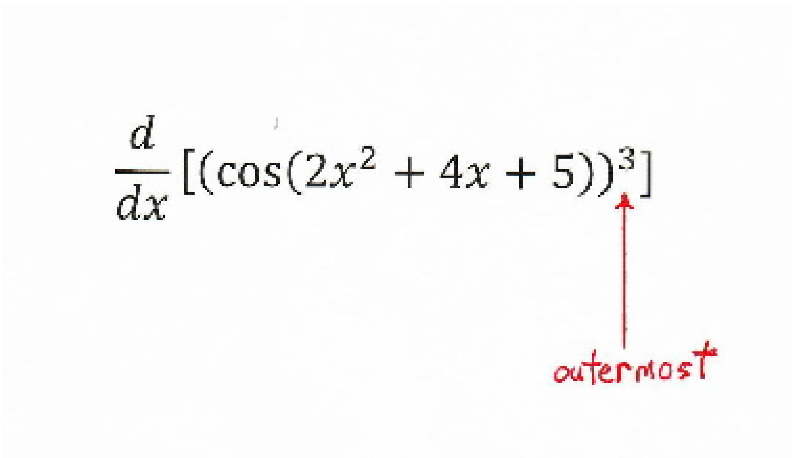
9. Compute: $\frac{d}{dx}\left[\cos^3\left(2x^2+4x+5\right)\right]=$

Let's rewrite this:

$\frac{d}{dx}\left[\left(\cos\left(2x^2+4x+5\right)\right)^3\right]$

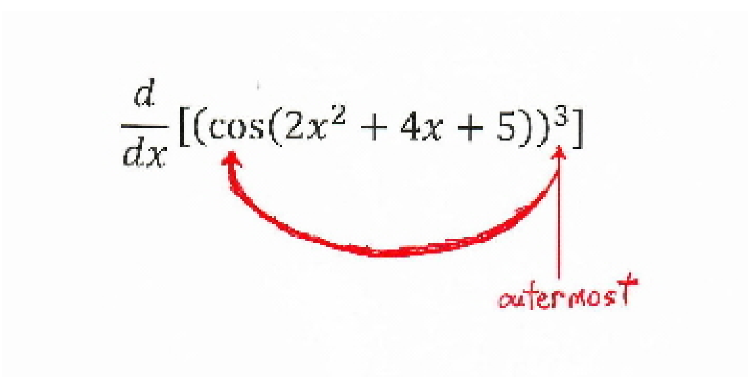
This is the composition of *three* functions.

Differentiate the outermost function and evaluate it at everything inside



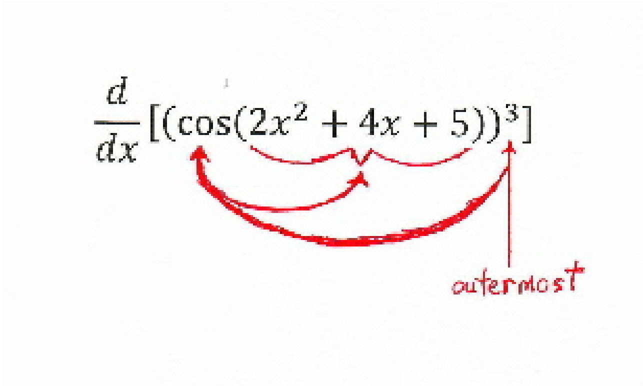
This yields: $3\left(\cos\left(2x^2+4x+5\right)\right)^2$

Next: Multiply by the derivative of the next outermost function and evaluate that at everything inside of it.



This yields: $3\left(\cos\left(2x^2+4x+5\right)\right)^2\cdot\left(-\sin\left(2x^2+4x+5\right)\right)$

Finally: Multiply by the derivative of the innermost function.



This yields: $3\left(\cos\left(2x^2+4x+5\right)\right)^2 \cdot \left(-\sin\left(2x^2+4x+5\right)\right) \cdot (4x+4)$

i.e., $\frac{d}{dx}\left[\cos^3\left(2x^2+4x+5\right)\right] = -3\left(\cos\left(2x^2+4x+5\right)\right)^2 \cdot \sin\left(2x^2+4x+5\right) \cdot (4x+4)$

Alternatively:

Re-Write!

$\frac{d}{dx}\left[\cos^3\left(2x^2+4x+5\right)\right] = \frac{d}{dx}\left[\left(\cos\left(2x^2+4x+5\right)\right)^3\right]$

In the broadest sense, this is *the derivative of a function raised to a power*

$$\frac{d}{dx}\left[\left(\cos\left(2x^2+4x+5\right)\right)^3\right] = \underbrace{3\left(\cos\left(2x^2+4x+5\right)\right)^2}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\cos\left(2x^2+4x+5\right)\right]\right)}_{\substack{\text{derivative}\\ \text{of inner}}}$$

$$= 3\left(\cos\left(2x^2+4x+5\right)\right)^2 \cdot \underbrace{\left[-\sin\left(2x^2+4x+5\right) \cdot (4x+4)\right]}_{\substack{\text{Chain}\\ \text{Rule}}}$$

i.e., $\frac{d}{dx}\left[\cos^3\left(2x^2+4x+5\right)\right] = -3\left(\cos\left(2x^2+4x+5\right)\right)^2 \cdot \sin\left(2x^2+4x+5\right) \cdot (4x+4)$

10. Given that $3y^2 + 6x^3y^5 + 2x^5 = \tan(y)$, compute $\frac{dy}{dx}$

i. Differentiate both sides w.r.t. x

$$\frac{d}{dx}\left[3y^2 + \underbrace{6x^3}_{1^{\text{st}}}\underbrace{y^5}_{2^{\text{nd}}} + 2x^5\right] = \frac{d}{dx}[\tan(y)]$$

$$\Rightarrow 6y\frac{dy}{dx} + \left(\underbrace{18x^2}_{1^{\text{st}}\text{ prime}} \cdot \underbrace{y^5}_{2^{\text{nd}}} + \underbrace{5y^4\frac{dy}{dx}}_{2^{\text{nd}}\text{ prime}} \cdot \underbrace{6x^3}_{1^{\text{st}}} + 10x^4\right) = \sec^2(y) \cdot \frac{dy}{dx}$$

Simplifying, we have:

$$6y\frac{dy}{dx} + 18x^2y^5 + 30x^3y^4\frac{dy}{dx} + 10x^4 = \sec^2(y)\frac{dy}{dx}$$

ii. Solve algebraically for $\frac{dy}{dx}$

a. Get $\frac{dy}{dx}$ terms on left side, all other terms on right side

$$\Rightarrow 6y\frac{dy}{dx} + 30x^3y^4\frac{dy}{dx} - \sec^2(y)\frac{dy}{dx} = -18x^2y^5 - 10x^4$$

b. Factor out $\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx}\left(6y + 30x^3y^4 - \sec^2(y)\right) = -18x^2y^5 - 10x^4$$

c. Divide both sides by the cofactor of $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-18x^2y^5 - 10x^4}{6y+30x^3y^4-\sec^2(y)} = -\frac{18x^2y^5 + 10x^4}{6y+30x^3y^4-\sec^2(y)}$$

$$\boxed{\frac{dy}{dx} = \frac{-18x^2y^5 - 10x^4}{6y+30x^3y^4-\sec^2(y)} = -\frac{18x^2y^5 + 10x^4}{6y+30x^3y^4-\sec^2(y)}}$$

11. Given that $f(x) = 2x^2 + 6x + 7$, compute $f'(x)$ **using the definition of derivative.** (i.e., using the "limit process.")

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[2(x+\Delta x)^2 + 6(x+\Delta x) + 7\right] - \left[2x^2+6x+7\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[2\left(x^2+2x\Delta x+\Delta x^2\right)+6(x+\Delta x)+7\right]-\left[2x^2+6x+7\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[2x^2+4x\Delta x+2\Delta x^2+6x+6\Delta x+7\right]-\left[2x^2+6x+7\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{4x\Delta x+2\Delta x^2+6\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(4x+2\Delta x+6)}{\Delta x} = \lim_{\Delta x \to 0} (4x + 2\Delta x + 6) = 4x + 2(0) + 6 = 4x + 6$$

i.e., $f'(x) = 4x + 6$

Extra (Wow! 10 Points)

Given that $L'(x) = \frac{2x+6}{x^2+6x+5}$ (i.e., $\frac{d}{dx}[L(x)] = \frac{2x+6}{x^2+6x+5}$); compute $\frac{d}{dx}[L(\tan(x))] =$

Outer:	$=$	$L(\quad)$
Deriv. of outer	$=$	$\frac{2(\quad)+6}{(\quad)^2+6(\quad)+5}$

$$\frac{d}{dx}\Bigg[\underset{\substack{\uparrow \\ \text{outer}}}{L}\Bigg(\underset{\substack{\uparrow \\ \text{inner}}}{\underbrace{\tan(x)}}\Bigg)\Bigg] = \underbrace{\frac{2(\tan(x))+6}{(\tan(x))^2+6(\tan(x))+5}}_{\substack{\text{Deriv of outer,} \\ \text{eval. at inner}}} \cdot \underbrace{\sec^2(x)}_{\substack{\text{deriv. of} \\ \text{inner}}} = \frac{(2\tan(x)+6)\sec^2(x)}{\tan^2(x)+6\tan(x)+5}$$

i.e., $\frac{d}{dx}[L(\tan(x))] = \frac{(2\tan(x)+6)\sec^2(x)}{\tan^2(x)+6\tan(x)+5}$