MTH 1126 - Test #2 - Solutions

FALL 2015

Pat Rossi Name ______________________

Instructions. Show CLEARLY how you arrive at your answers.

1. Compute the arclength of the graph of the function $f(x) = 6x^{\frac{3}{2}} + 5$ from the point $(0,5)$ and $(1,11)$.

   Use the formula: Arc Length $= \int_a^b \sqrt{1 + (f'(x))^2}dx$

   $f'(x) = 9x^{\frac{1}{2}}$

   $(f'(x))^2 = \left(9x^{\frac{1}{2}}\right)^2 = 81x$

   $\Rightarrow$Arc Length $= \int_a^b \sqrt{1 + (f'(x))^2}dx = \int_{x=0}^{x=1} \sqrt{1+81x}dx = \int_{x=0}^{x=1} \underbrace{(1+81x)^{\frac{1}{2}}}_{u^{\frac{1}{2}}} \underbrace{dx}_{\frac{1}{81}du}$

   $$\begin{array}{rrcl} & u & = & 1+81x \\ \Rightarrow & du & = & 81dx \\ \Rightarrow & \frac{1}{81}du & = & dx \\ & & & \text{When } x=0,\ u = 1+81(0) = 1 \\ & & & \text{When } x=1,\ u = 1+81(1) = 82 \end{array}$$

   $= \int_{u=1}^{u=82} u^{\frac{1}{2}} \frac{1}{81}du = \frac{1}{81}\int_{u=1}^{u=82} u^{\frac{1}{2}}du = \frac{1}{81}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=1}^{u=82} = \frac{2}{243}(82)^{\frac{3}{2}} - \frac{2}{243}(1)^{\frac{3}{2}}$

   i.e., Arclength $= \frac{2}{243}(82)^{\frac{3}{2}} - \frac{2}{243}(1)$

2. Use the "$f - g$" method to compute the area bounded by the graphs of $f(x) = x^2 - 1$ and $g(x) = x + 5$.

First, graph the functions and find the points of intersection.

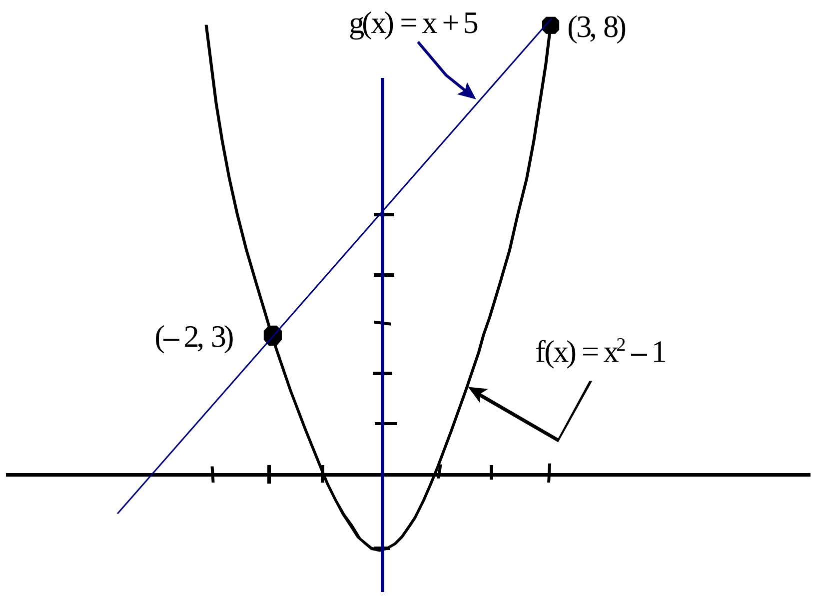
$y = x^2 - 1 = x + 5$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x+2)(x-3) = 0$

$x = -2$; $x = 3$

Points of intersection are $(-2, 3)$ and $(3, 8)$.



The bounded region spans the interval $[-2, 3]$ on the x-axis. Over this interval, $g(x) = x + 5$ is greater than $f(x) = x^2 - 1$. Hence the area is given by:

$\int_{-2}^{3} [(x+5) - (x^2 - 1)]\, dx = \int_{-2}^{3} (-x^2 + x + 6)\, dx = \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2 + 6x\right]_{-2}^{3}$

$= \left(-\frac{1}{3}(3)^3 + \frac{1}{2}(3)^2 + 6(3)\right) - \left(-\frac{1}{3}(-2)^3 + \frac{1}{2}(-2)^2 + 6(-2)\right) = \frac{125}{6}$

i.e., bounded area $= \frac{125}{6}$

3. Find the area bounded by the graphs of $f(x) = 4 - x^2$ and $g(x) = -2x + 4$. (Partition the appropriate interval, sketch the i$^{\text{th}}$ rectangle, build the Riemann Sum, derive the appropriate integral.)

Graph the functions and find the points of intersection.

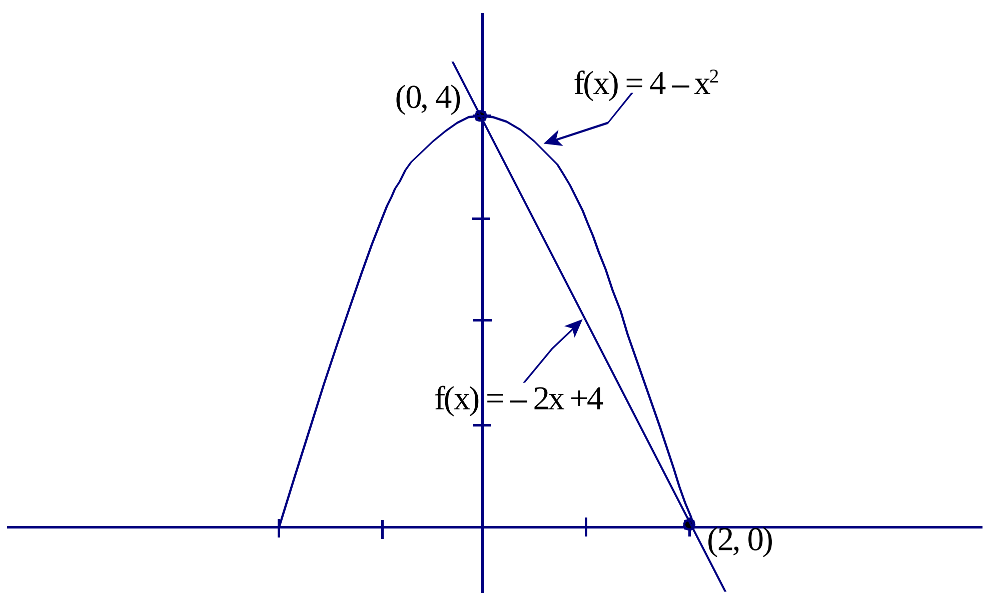
$y = 4 - x^2 = -2x + 4$

$\Rightarrow x^2 - 2x = 0$

$\Rightarrow x(x-2) = 0.$

$\Rightarrow x = 0$; and $x = 2$.

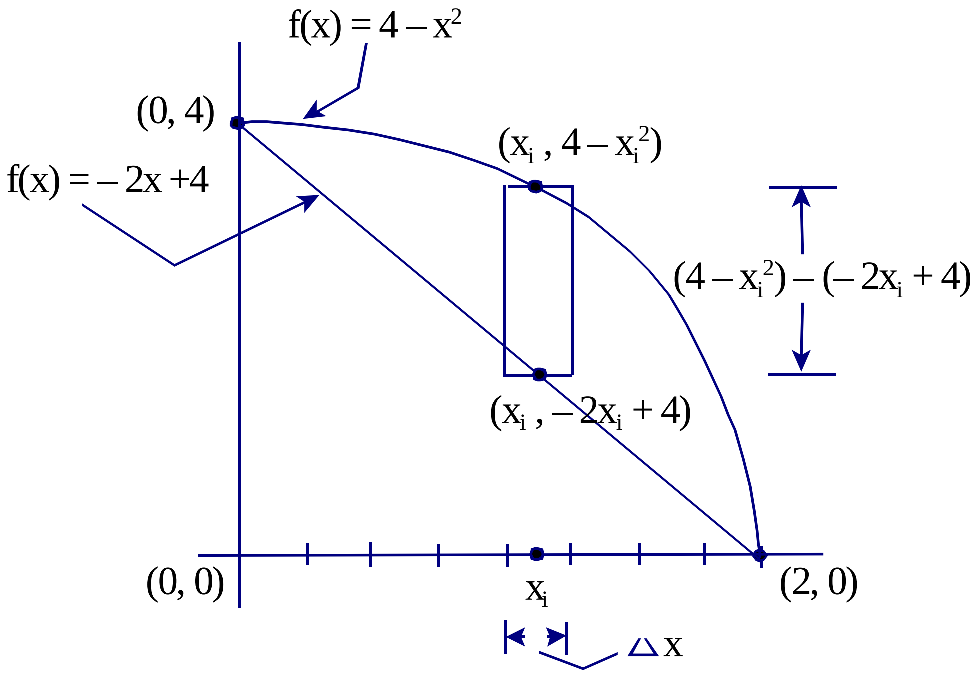
Points of intersection: $(0,4)$ and $(2,0)$.



The rectangles span the interval $[0,2]$ on the x-axis, so we will partition that interval into sub-intervals of length Δx.

The area of the i^{th}. rectangle is $\underbrace{\left((4 - x_i^2) - (-2x_i + 4)\right)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}} = (-x_i^2 + 2x_i)\,\Delta x$

(see below)



To approximate the area of the bounded region, we add the areas of the rectangles:

$A \approx \sum_{i=1}^{n} \left(-x_i^2 + 2x_i\right) \Delta x$

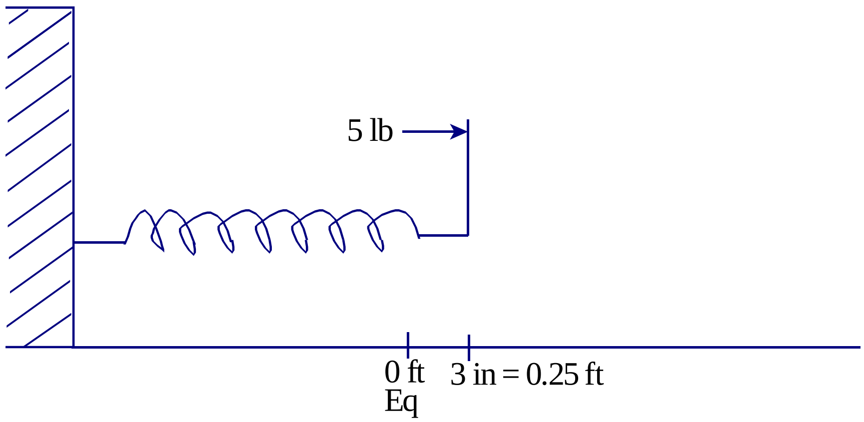
To get the exact area, we let $\Delta x \to 0$.

$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \left(-x_i^2 + 2x_i\right) \Delta x = \int_0^2 \left(-x^2 + 2x\right) dx = \left[-\frac{1}{3}x^3 + x^2\right]_0^2$

$= \left(-\frac{1}{3}(2)^3 + (2)^2\right) - \left(-\frac{1}{3}(0)^3 + (0)^2\right) = \frac{4}{3}$

i.e., bounded area $= \frac{4}{3}$

4. Five pounds of force is required to stretch a spring 3 inches past the point of equilibrium. How much work is done stretching the spring 12 inches past the point of equilibrium? (Partition the appropriate interval, compute F_i, build the Riemann Sum, derive the appropriate integral.)



First, we have to find the spring constant k, using the values $F = 5$ lb and

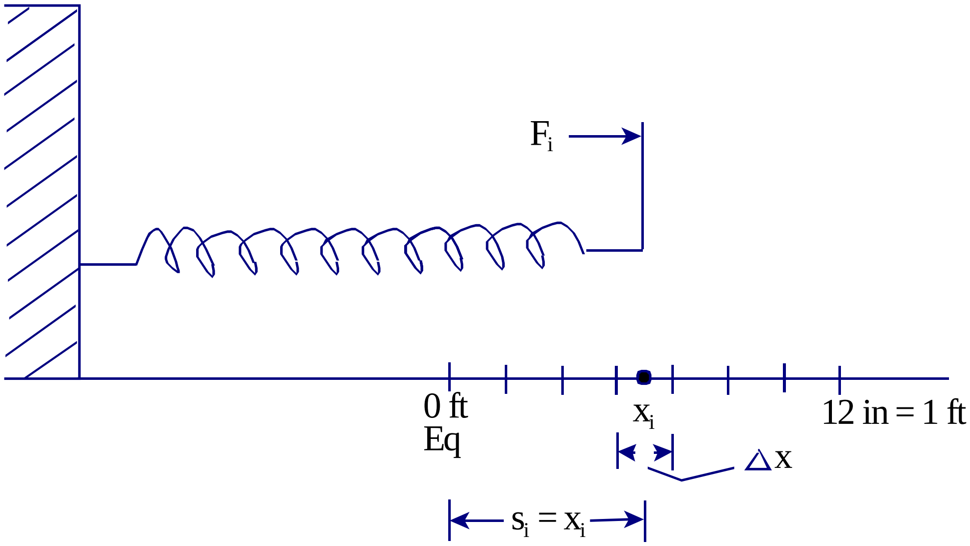
$s = 3$ inches $= \frac{1}{4}$ ft $= 0.25$ ft

From Hooke's Law $(F = ks)$ we have $k = \frac{F}{s} = \frac{5 \text{ lb}}{0.25 \text{ ft}} = 20 \ \frac{\text{lb}}{\text{ft}}$

i.e., $k = 20 \ \frac{\text{lb}}{\text{ft}}$

Hence, we have: $F = 20\frac{\text{lb}}{\text{ft}}s$

Next, partition the interval, over which the work is to be performed, and compute W_i, the work done stretching the spring from one side of the i^{th} sub-interval to the other side of the i^{th} sub-interval. (see below)



$W_i = F_i d_i$

Here, d_i is the distance over which the work W_i is performed

$d_i = \Delta x$

$F_i = k s_i = 20 \frac{\text{lb}}{\text{ft}} x_i$

Hence, $W_i = F_i d_i = 20 \frac{\text{lb}}{\text{ft}} x_i \Delta x$

i.e., $W_i = 20 \frac{\text{lb}}{\text{ft}} x_i \Delta x$

The total work, W_T, is approximately the sum of the work done stretching the spring across each sub-interval.

$W_T \approx \sum_{i=1}^{n} 20 \frac{\text{lb}}{\text{ft}} x_i \Delta x$

$W_T = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 20 \frac{\text{lb}}{\text{ft}} x_i \Delta x = \int_{0 \text{ ft}}^{1 \text{ ft}} 20 \frac{\text{lb}}{\text{ft}} x \, dx = 20 \frac{\text{lb}}{\text{ft}} \int_{0 \text{ ft}}^{1 \text{ ft}} x \, dx = 20 \frac{\text{lb}}{\text{ft}} \left[\frac{x^2}{2} \right]_{0 \text{ ft}}^{1 \text{ ft}}$

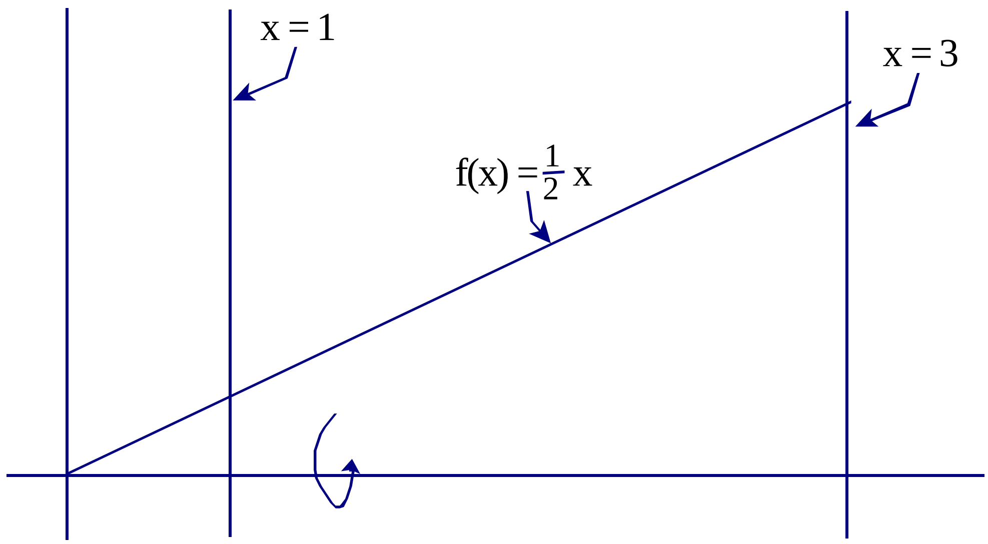
$= 20 \frac{\text{lb}}{\text{ft}} \left[\left(\frac{(1 \text{ ft})^2}{2} \right) - \left(\frac{(0 \text{ ft})^2}{2} \right) \right] = 10 \text{ lb ft}$

i.e., $W_T = 10$ lb ft

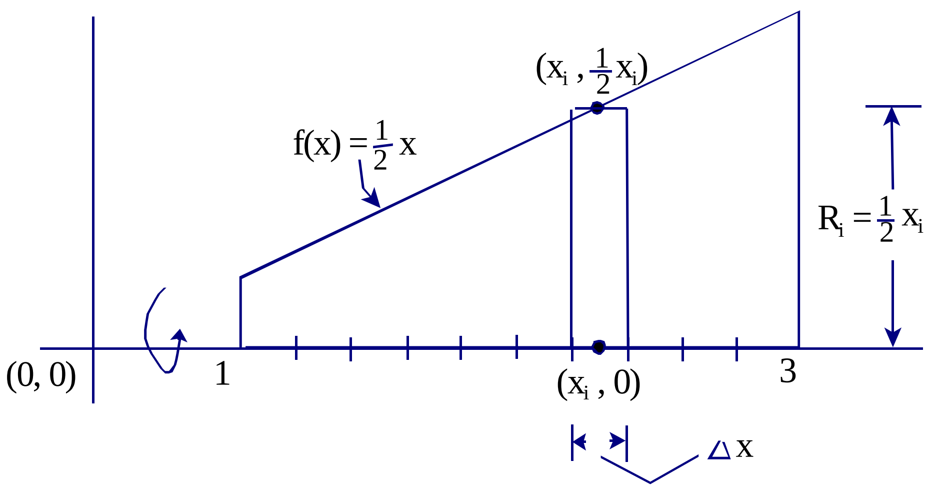
5. Use the "disc method" to compute the volume of the solid of revolution generated by revolving the region bounded by the graphs of $f(x) = \frac{1}{2}x$, $x = 1$, $x = 3$, and the x-axis, about the x-axis. (For your information: the equation of the x-axis is $y = 0$.)

Use the "five step method" (partition the interval, sketch the i^{th} rectangle, form the sum, take the limit)

i. First, graph the bounded area.



ii. Sketch a rectangle perpendicular (perpen-"disc"-ular) to the axis of revolution and partition the interval spanned by the rectangles.



iii. Revolve the i^{th} rectangle about the axis of revolution.

Vol. of i$^{\text{th}}$ disc $= \pi R_i^2 \Delta x = \pi \left(\frac{1}{2}x_i\right)^2 \Delta x = \pi \left(\frac{1}{4}x_i^2\right) \Delta x = \frac{\pi}{4}x_i^2 \Delta x$

iv. Approximate the volume of the solid of revolution by adding up the volumes of the discs

$Vol \approx \sum_{i=1}^{n} \frac{\pi}{4} x_i^2 \Delta x$

v. Let $\Delta x \to 0$

$Vol \approx \lim_{\Delta x \to 0} \sum_{i=1}^{n} \frac{\pi}{4} x_i^2 \Delta x = \int_{x=1}^{x=3} \frac{\pi}{4} x^2 dx$

$= \frac{\pi}{4} \left[\frac{x^3}{3}\right]_{x=1}^{x=3} = \frac{\pi}{4}\frac{(3)^3}{3} - \frac{\pi}{4}\frac{(1)^3}{3} = \frac{13\pi}{6}$

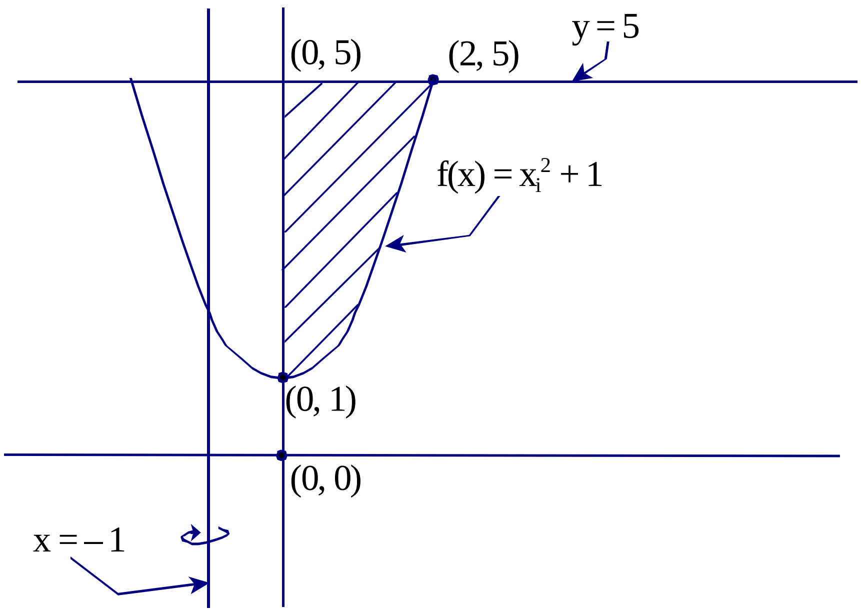
i.e., Volume $= \frac{13\pi}{6}$

6. Use the "shell method" to compute the volume of the solid of revolution generated by revolving the region described below about the line $x = -1$.

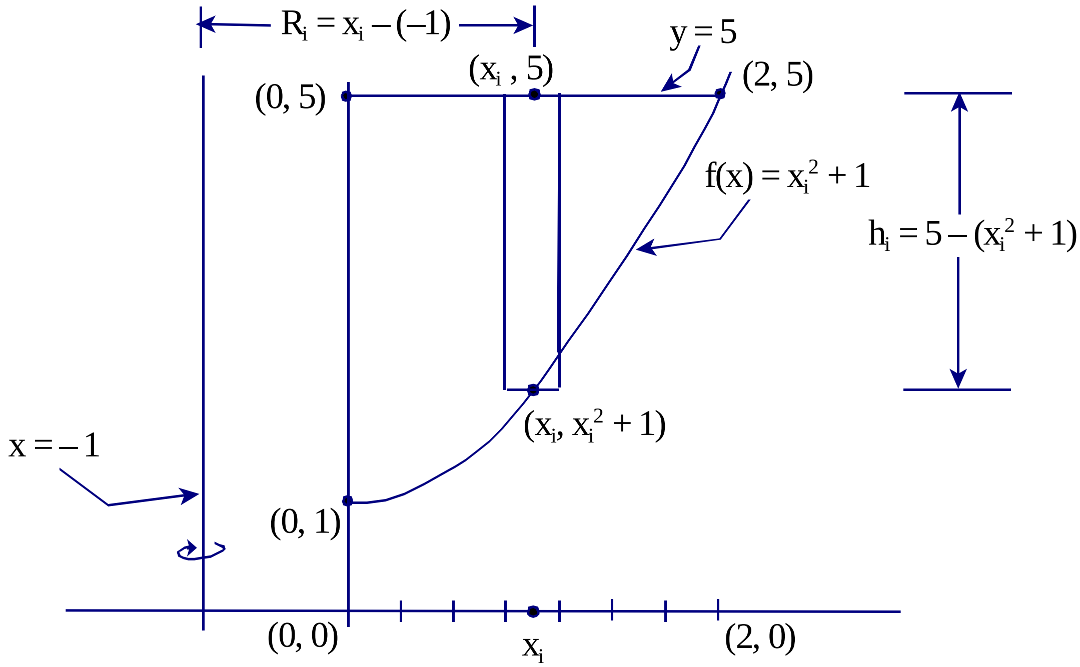
   The region lies to the right of the y-axis and is bounded by the graph $f(x) = x^2 + 1$, the y-axis, and the line $y = 5$.

   Use the "five step method" (partition the interval, sketch the i$^{\text{th}}$ rectangle, form the sum, take the limit)

   i. First, graph the bounded area.



   ii. Sketch a rectangle parallel to the axis of revolution ("shell - parallel"), and partition the interval spanned by the rectangles



iii. Revolve the i$^{\text{th}}$ rectangle about the axis of revolution to form the i$^{\text{th}}$ shell.

Vol. i$^{\text{th}}$ shell $= 2\pi R_i h_i \Delta x = 2\pi \left(x_i - (-1)\right)\left(5 - \left(x_i^2 + 1\right)\right)\Delta x$

$$= 2\pi \left(x_i + 1\right)\left(4 - x_i^2\right)\Delta x = 2\pi \left(4x_i - x_i^3 + 4 - x_i^2\right)\Delta x$$

iv. Approximate the volume of the solid of revolution by adding the volumes of the shells.

$Vol \approx \sum_{i=1}^{n} 2\pi \left(4x_i - x_i^3 + 4 - x_i^2\right)\Delta x$

v. Let $\Delta x \to 0$

$Vol = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 2\pi \left(4x_i - x_i^3 + 4 - x_i^2\right)\Delta x = \int_{x=0}^{x=2} 2\pi \left(4x - x^3 + 4 - x^2\right) dx$

$= 2\pi \left[2x^2 - \frac{1}{4}x^4 + 4x - \frac{1}{3}x^3\right]_{x=0}^{x=2}$

$= 2\pi \left(2\,(2)^2 - \frac{1}{4}\,(2)^4 + 4\,(2) - \frac{1}{3}\,(2)^3\right) - 2\pi \left(2\,(0)^2 - \frac{1}{4}\,(0)^4 + 4\,(0) - \frac{1}{3}\,(0)^3\right)$

$= \frac{56}{3}\pi$

i.e., $Vol = \frac{56\pi}{3}$