Proofs Involving Sets #6 (Miscellaneous Exercises) - Solutions

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Instructions. Prove the following:

1. $U^c = \emptyset$

   Proof. Observe: $\forall x$, $x \in U$ (By definition of universe)

   $\Rightarrow \forall x$, $x \notin U^c$

   $\Rightarrow U^c = \emptyset$, (Because by definition of empty set, $\emptyset$ is the unique set such that $\forall x$, $x \notin \emptyset$) ■

2. $\emptyset^c = U$

   Proof. Observe: $\forall x$, $x \notin \emptyset$ (By definition of empty set)

   $\Rightarrow \forall x$, $x \in \emptyset^c$

   $\Rightarrow \emptyset^c = U$ (Because by definition of universe, U is the unique set such that $\forall x$, $x \in U$) ■

3. $(A^c)^c = A$

   Proof. We must show that:

   i. $(A^c)^c \subseteq A$

   and

   ii. $A \subseteq (A^c)^c$

   | i. $(A^c)^c \subseteq A$ |

   Let $x \in (A^c)^c$

   $\Rightarrow x \notin (A^c)$

   $\Rightarrow x \in A$

   i.e., $x \in (A^c)^c \Rightarrow x \in A$

   Hence, $A \subseteq (A^c)^c$

   | ii. $A \subseteq (A^c)^c$ |

   Let $x \in A$

   $\Rightarrow x \notin (A^c)$

   $\Rightarrow x \in (A^c)^c$

   i.e., $x \in A \Rightarrow x \in (A^c)^c$

   Hence, $A \subseteq (A^c)^c$ ■

4. $(A \cup B) \cup C = A \cup (B \cup C)$

Proof. We must show that:

i. $(A \cup B) \cup C \subseteq A \cup (B \cup C)$

and

ii. $A \cup (B \cup C) \subseteq (A \cup B) \cup C$

i. $(A \cup B) \cup C \subseteq A \cup (B \cup C)$

Let $x \in (A \cup B) \cup C$

$\Rightarrow x \in (A \cup B)$ or $x \in C$

$\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$

(Note that the implication of the preceeding statement is that x is an element of at least one of the sets)

$\Rightarrow x \in A$ or $(x \in B$ or $x \in C)$ (Because the implication of *this* statement is *also* that x is an element of at least one of the sets)

$\Rightarrow x \in A$ or $(x \in B \cup C)$

$\Rightarrow x \in A \cup (B \cup C)$

i.e., $x \in (A \cup B) \cup C \Rightarrow x \in A \cup (B \cup C)$

Hence, $(A \cup B) \cup C \subseteq A \cup (B \cup C)$

ii. $A \cup (B \cup C) \subseteq (A \cup B) \cup C$

Let $x \in A \cup (B \cup C)$

$\Rightarrow x \in A$ or $(x \in B \cup C)$

$\Rightarrow x \in A$ or $(x \in B$ or $x \in C)$ (Note that the implication here is also that x is an element of at least one of the sets)

$\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$ (Note that the implication is that x is an element of at least one of the sets)

$\Rightarrow x \in (A \cup B)$ or $x \in C$

$\Rightarrow x \in (A \cup B) \cup C$

i.e., $x \in A \cup (B \cup C) \Rightarrow x \in (A \cup B) \cup C$

Hence, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$ ■

5. $(A \cap B) \cap C = A \cap (B \cap C)$

Proof. We must show that:

i. $(A \cap B) \cap C \subseteq A \cap (B \cap C)$

and

ii. $A \cap (B \cap C) \subseteq (A \cap B) \cap C$

i. $(A \cap B) \cap C \subseteq A \cap (B \cap C)$

(a) Let $x \in (A \cap B) \cap C$

$\Rightarrow x \in (A \cap B)$ and $x \in C$

$\Rightarrow (x \in A$ and $x \in B)$ and $x \in C$

(Note that the implication of the preceeding statement is that x is an element of all three sets)

$\Rightarrow x \in A$ and $(x \in B$ and $x \in C)$ (Because the implication of *this* statement is *also* that x is an element of all three sets)

$\Rightarrow x \in A$ and $(x \in B \cap C)$

$\Rightarrow x \in A \cap (B \cap C)$

i.e., $x \in (A \cap B) \cap C \Rightarrow x \in A \cap (B \cap C)$

Hence, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$

ii. $A \cap (B \cap C) \subseteq (A \cap B) \cap C$

Let $x \in A \cap (B \cap C)$

$\Rightarrow x \in A$ and $(x \in B \cap C)$

$\Rightarrow x \in A$ and $(x \in B$ and $x \in C)$

(Note that the implication of the preceeding statement is that x is an element of all three sets)

$\Rightarrow (x \in A$ and $x \in B)$ and $x \in C$ (Note that the implication this statement is *also* that x is an element of all three sets)

$\Rightarrow x \in (A \cap B)$ and $x \in C$

$\Rightarrow x \in (A \cap B) \cap C$

i.e., $x \in A \cap (B \cap C) \Rightarrow x \in (A \cap B) \cap C$

Hence, $A \cap (B \cap C) \subseteq (A \cap B) \cap C$ ■

6. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. This proof is the same as the proof of *Lemma 1* on *Proofs Involving Sets #8* ■