# MTH 1126 - Test #2

SPRING 2023

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**Instructions.** Show CLEARLY how you arrive at your answers.

Do Exercises #1, 2. From Exercises #3-5, select two.

1. Use the "$f - g$" method to compute the area bounded by the graphs of $f(x) = x^2 - 5$ and $g(x) = 3 - x^2$.

   First, graph the functions and find the points of intersection.

   $y = x^2 - 5 = 3 - x^2$

   i.e., $x^2 - 5 = 3 - x^2$

   $\Rightarrow 2x^2 - 5 = 3$

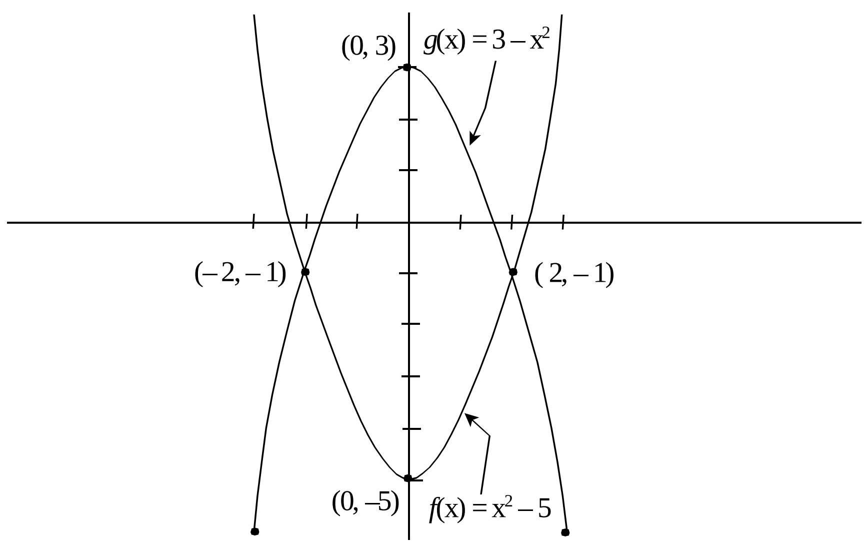
   $\Rightarrow 2x^2 - 8 = 0$

   $\Rightarrow x^2 - 4 = 0$

   $\Rightarrow (x+2)(x-2) = 0$

   $x = -2$; $x = 2$

   Points of intersection are $(-2, -1)$ and $(2, -1)$.




   The bounded region spans the interval $[-2, 2]$ on the $x$-axis. Over this interval, $g(x) = 3 - x^2$ is greater than $f(x) = x^2 - 5$. Hence the area is given by:

$$A = \int_{-2}^{2} \left( (3 - x^2) - (x^2 - 5) \right) dx = \int_{-2}^{2} (8 - 2x^2)\, dx = \left[ 8x - \tfrac{2}{3}x^3 \right]_{-2}^{2}$$

$$= \left( 8\,(2) - \tfrac{2}{3}\,(2)^3 \right) - \left( 8\,(-2) - \tfrac{2}{3}\,(-2)^3 \right) = \tfrac{64}{3}$$

i.e., bounded area $= \frac{64}{3}$

2. Find the area bounded by the graphs of $f(x) = 3x$ and $g(x) = x^2$. (Partition the appropriate interval, sketch the i$^{\text{th}}$ rectangle, build the Riemann Sum, derive the appropriate integral.)

Graph the functions and find the points of intersection.

To find the points of intersection, set the y-coordinates equal to one another and solve for x.

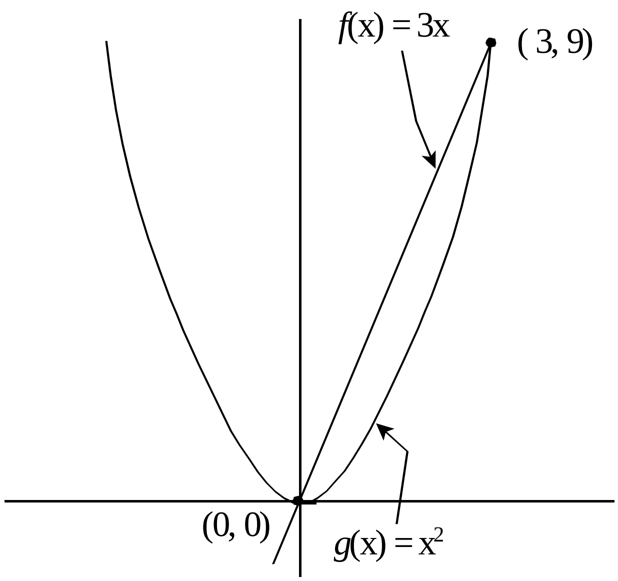
$y = 3x = x^2$

$\Rightarrow 3x - x^2 = 0$

$\Rightarrow x(3 - x) = 0.$

$\Rightarrow x = 0$; and $x = 3$.

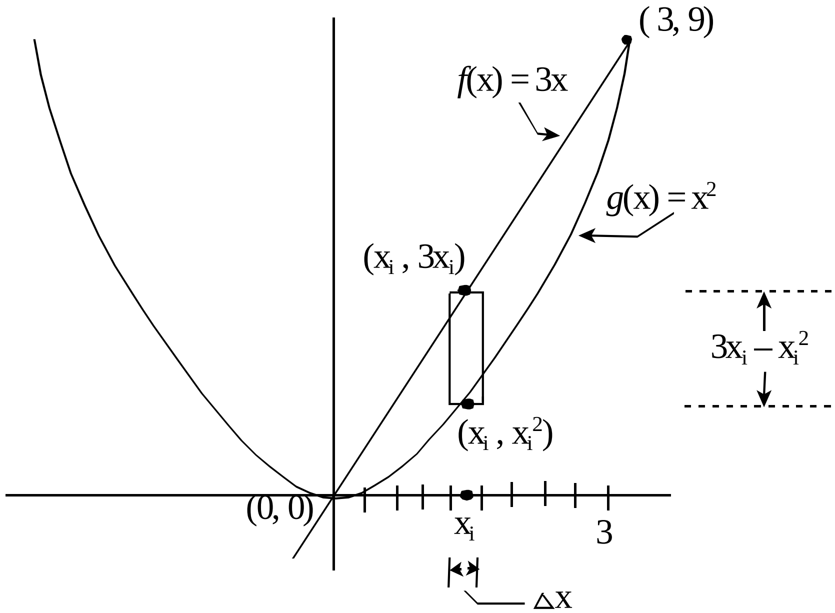
Points of intersection: $(0, 0)$ and $(3, 9)$.




The rectangles span the interval $[0, 3]$ on the $x$-axis, so we will partition that interval into sub-intervals of length $\Delta x$.

The area of the $i^{th}$. rectangle is $\underbrace{\left((x_i + 2) - \left(x_i^2 - 4\right)\right)}_{\text{height}} \cdot \underbrace{\Delta x}_{\text{width}} = \left(-x_i^2 + x_i + 6\right) \Delta x$

(see below)




To approximate the area of the bounded region, we add the areas of the rectangles:

$A \approx \sum_{i=1}^{n} (3x_i - x_i^2)\,\Delta x$

To get the exact area, we let $\Delta x \to 0$.

$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} (3x_i - x_i^2)\,\Delta x = \int_0^3 (3x - x^2)\,dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3$

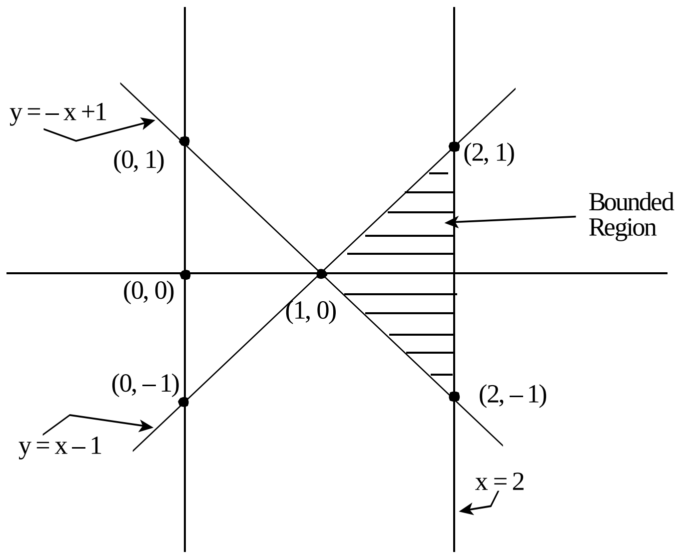
$= \left(\frac{3}{2}(3)^2 - \frac{1}{3}(3)^3\right) - \left(\frac{3}{2}(0)^2 - \frac{1}{3}(0)^3\right) = \frac{9}{2}$

i.e., bounded area $= \frac{9}{2}$

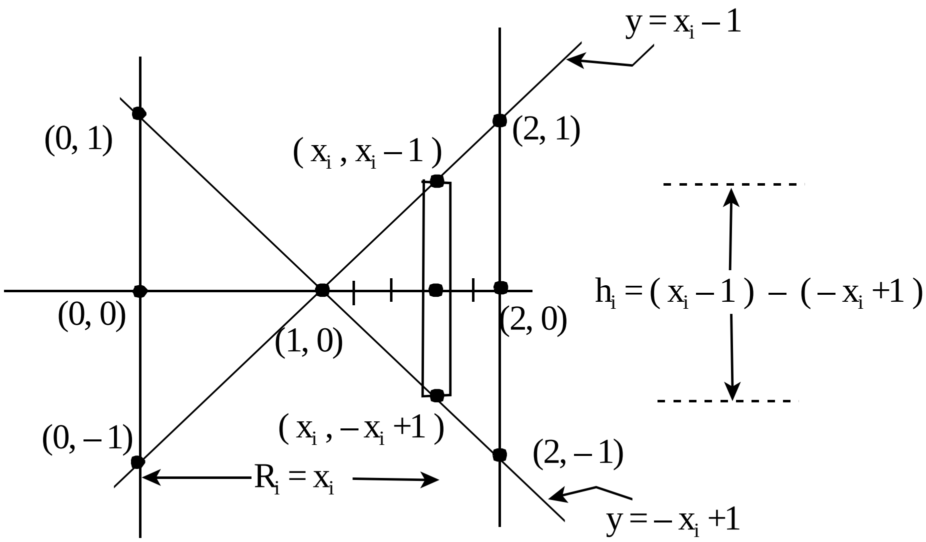
3. Use the "shell method" to compute the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = x - 1$, $y = -x + 1$, and $x = 2$, about the $y$-axis. (For your information: the equation of the $y$-axis is $x = 0$.)

   Use the "five step method" (partition the interval, sketch the $i^{th}$ rectangle, form the sum, take the limit)

   i) First, we'll graph the bounded region.



ii) Next, we sketch a rectangle of width $\Delta x$ parallel ("shell-parallel") to the axis of revolution, and we partition the interval spanned by the rectangles.




iii) Revolve the i[th] rectangle about the axis of revolution and compute the volume of the i[th] shell, $Vol_i$

$$Vol_i = 2\pi R_i h_i \Delta x = 2\pi x_i \left[(x_i - 1) - (-x_i + 1)\right] \Delta x = 2\pi x_i \left(2x_i - 2\right) \Delta x$$

$$= 4\pi x_i \left(x_i - 1\right) \Delta x = 4\pi \left(x_i^2 - x_i\right) \Delta x$$

iv) Approximate the volume of the solid by adding up the volumes of the shells

$$Vol \approx \sum_{i=1}^{n} 4\pi \left(x_i^2 - x_i\right) \Delta x$$

v) Let $\Delta x \to 0$

$$Vol = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 4\pi \left(x_i^2 - x_i\right) \Delta x = \int_{x=1}^{x=2} 4\pi \left(x^2 - x\right) dx = 4\pi \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{x=1}^{x=2}$$

$$= 4\pi \left[\left(\frac{(2)^3}{3} - \frac{(2)^2}{2}\right) - \left(\frac{(1)^3}{3} - \frac{(1)^2}{2}\right)\right] = \frac{10}{3}\pi$$

$$\boxed{Vol = \frac{10}{3}\pi}$$

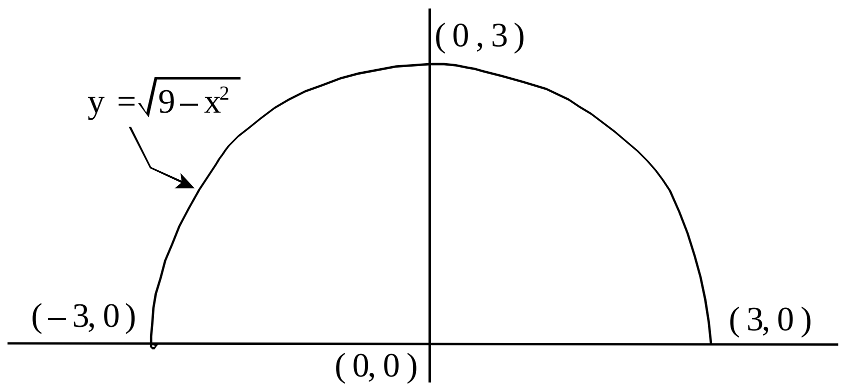
4. Use the "disc method" to compute the volume of the solid of revolution generated by revolving the region described below about the $x$-axis.

The region lies above the $x$-axis and is bounded by the graph $y = \sqrt{9 - x^2}$, and the $x$-axis.

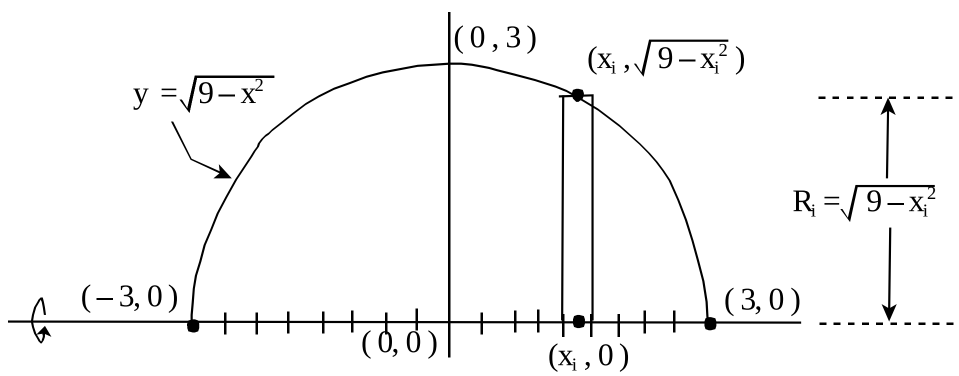
Use the "five step method" (partition the interval, sketch the i$^{\text{th}}$ rectangle, form the sum, take the limit)

i) First, we'll graph the bounded region.

**Note:** $y = \sqrt{9 - x^2}$ (same as $y = \sqrt{3^2 - x^2}$) is the equation of the semi-circle of radius 3, centered at the origin, that lies above the $x$-axis.




ii) Next, we sketch a rectangle of width $\Delta x$ perpendicular (perpen-"disc"-ular) to the axis of revolution, and we partition the interval spanned by the rectangles.




iii) Revolve the i$^{\text{th}}$ rectangle about the axis of revolution and compute the volume of the i$^{\text{th}}$ disc, $Vol_i$

$$Vol_i = \pi R_i^2 \Delta x = \pi \left(\sqrt{9 - x_i^2}\right)^2 \Delta x = \pi \left(9 - x_i^2\right) \Delta x$$

iv) Approximate the volume of the solid by adding up the volumes of the discs

$$Vol \approx \pi\left(9 - x_i^2\right)\Delta x$$

v) Let $\Delta x \to 0$

$$Vol = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi\left(9 - x_i^2\right)\Delta x = \int_{x=-3}^{x=3} \pi\left(9 - x^2\right)dx$$

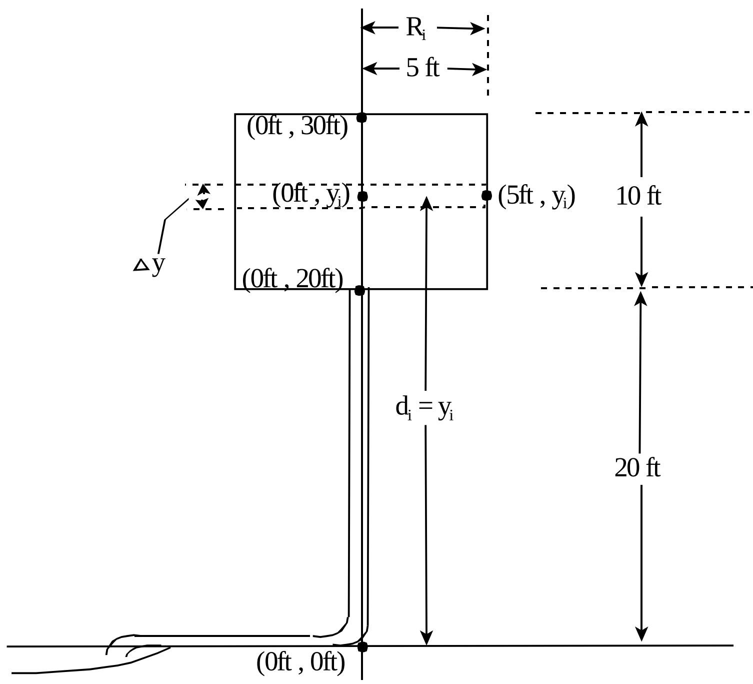
$$= \pi\left[9x - \frac{1}{3}x^3\right]_{x=-3}^{x=3} = \pi\left[9\left(3\right) - \frac{1}{3}\left(3\right)^3\right] - \pi\left[9\left(-3\right) - \frac{1}{3}\left(-3\right)^3\right]$$

$$= \pi\left[\left(27 - 9\right)\right] - \pi\left[\left(-27 + 9\right)\right] = 36\pi$$

$$\boxed{Vol = 36\pi}$$

5. Compute the work done in filling the reservoir of a water tower, though a hole in the bottom of the reservoir. The reservoir is a cylindrical tank of height 10 ft and radius 5 ft. The base of the reservoir is 20 ft above the level of the pond from which the water is pumped. (Assume that water weighs $\rho = 100\frac{\text{lbs}}{\text{ft}^3}$)

   We will partition the water in the tank into horizontal slices of width $\triangle y$ and assume that $\triangle y$ is so small that, for all practical purposes, every molecule in the each slice is exactly the same distance from ground level. Namely $d_i = y_i$




   Note that the volume of each slice is exactly the same. $V_i =$ (cross-sectional area) (thickness) $= \pi R_i^2 \triangle y$

   The weight of the $i^{th}$ slice $=$ (volume) (weight per unit volume) $= V_i \cdot \rho$

   The force $F_i$ needed to move the $i^{th}$ slice upward is the same as the weight of the $i^{th}$ slice

   Therefore $F_i = w_i = V_i \cdot \rho$

   Therefore, $W_i$ the work done in pumping the $i^{th}$ slice to its final height above ground, is the applied **force** times the **distance** over which the force is applied: $W_i = F_i \cdot d_i$

   The total work done filling the reservior is approximately the sum of the work done pumping each slice to its final height: $W \approx \sum_{i=1}^{n} F_i \cdot d_i$

   $$W \approx \sum_{i=1}^{n} F_i \cdot d_i = \sum_{i=1}^{n} V_i \cdot \rho \cdot d_i = \sum_{i=1}^{n} (\pi R_i^2 \triangle y) \cdot \rho \cdot d_i = \sum_{i=1}^{n} (\pi R_i^2 \triangle y) \cdot \rho \cdot y_i$$

   $$= \sum_{i=1}^{n} \left(\pi\, (5\text{ft})^2 \triangle y\right) \cdot \rho \cdot y_i = \sum_{i=1}^{n} \left(\pi 25\text{ft}^2 \triangle y\right) \cdot \rho \cdot y_i = 25\text{ft}^2 \rho \pi \sum_{i=1}^{n} y_i \triangle y$$

i.e., $W \approx 25\text{ft}^2\rho\pi\sum_{i=1}^{n} y_i \triangle y$

To get the exact amount of work done, we let $\triangle y \to 0$

$$W = \lim_{\triangle y \to 0} 25\text{ft}^2\rho\pi\sum_{i=1}^{n} y_i \triangle y = 25\text{ft}^2\rho\pi\int_{20\text{ft}}^{30\text{ft}} y dy = 25\text{ft}^2\rho\pi\left[\frac{y^2}{2}\right]_{y=20\text{ft}}^{y=30\text{ft}}$$

$$= 25\text{ft}^2\rho\pi\left[\left(\frac{(30\text{ft})^2}{2}\right) - \left(\frac{(20\text{ft})^2}{2}\right)\right] = 25\text{ft}^2\rho\pi\left[\left(\frac{900\text{ft}^2}{2}\right) - \left(\frac{400\text{ft}^2}{2}\right)\right]$$

$$= 25\text{ft}^2\rho\pi\left[\frac{500\text{ft}^2}{2}\right] = 25\text{ft}^2\rho\pi\left[250\text{ft}^2\right] = 6250\text{ft}^4\rho\pi = 6250\text{ft}^4\left(100\frac{\text{lbs}}{\text{ft}^3}\right)\pi$$

$$= 625,000\pi\text{ft lbs}$$

$$\boxed{w = 625,000\pi\text{ft lbs}}$$