MTH 3311 – Test #2 - Part #3 - Solutions

Spring 2020

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The supply and demand of a commodity are given in thousands of units by $S = 48 - 24e^{-2t} + 16p(t) + 10p'(t)$ and $D = 240 - 8p(t) - 2p'(t)$, respectively. At $t = 0$, the price of the commodity is 12 units.

a) Find the price at any later time and obtain its graph.

b) Determine whether there is price stability and determine the equilibrium price (if it exists).

Equating supply and demand, we have:

$48 - 24e^{-2t} + 16p(t) + 10p'(t) = 240 - 8p(t) - 2p'(t)$

$\Rightarrow 12p'(t) + 24p(t) = 192 + 24e^{-2t}$

$\Rightarrow p'(t) + 2p(t) = 16 + 2e^{-2t}$

$\Rightarrow p'(t) + \underbrace{2}_{P(t)} p(t) = \underbrace{16 + 2e^{-2t}}_{Q(t)}$

Our integrating factor is $e^{\int P(t)dt} = e^{\int 2dt} = e^{2t}$

Multiplying both sides by the integrating factor, e^{2t}, we have:

$e^{2t}p'(t) + 2e^{2t}p(t) = 16e^{2t} + 2$

$\Rightarrow \frac{d}{dt}\left[e^{2t}p(t)\right] = 16e^{2t} + 2$ Integrating, we have:

$\Rightarrow \int \left(\frac{d}{dt}\left[e^{2t}p(t)\right]\right) dt = \int \left(16e^{2t} + 2\right) dt$

$\Rightarrow e^{2t}p(t) = 16\left(\frac{1}{2}\right)e^{2t} + 2t + C = 8e^{2t} + 2t + C$

i.e., $e^{2t}p(t) = 8e^{2t} + 2t + C$

$\Rightarrow p(t) = 8 + 2te^{-2t} + e^{-2t}C$

To find the constant C, we use our initial condition $p(0) = 12$ (Because "At $t = 0$, the price of the commodity is 12 units.")

$\Rightarrow 12 = p(0) = 8 + 2(0)e^{-2(0)} + e^{-2(0)}C = 8 + C$

i.e., $12 = 8 + C$

i.e., $4 = C$

Hence, $p(t) = 8 + 2te^{-2t} + 4e^{-2t}$ is the price at any time t.

To graph the function, let's consider the derivative.

$p'(t) = 2e^{-2t} - 4te^{-2t} - 8e^{-2t} = -4te^{-2t} - 6e^{-2t}$

i.e. $p'(t) = -4te^{-2t} - 6e^{-2t}$

Note that $p'(t) < 0$ for all values of $t > 0$, since $e^{\text{ham sandwich}}$ is always positive.

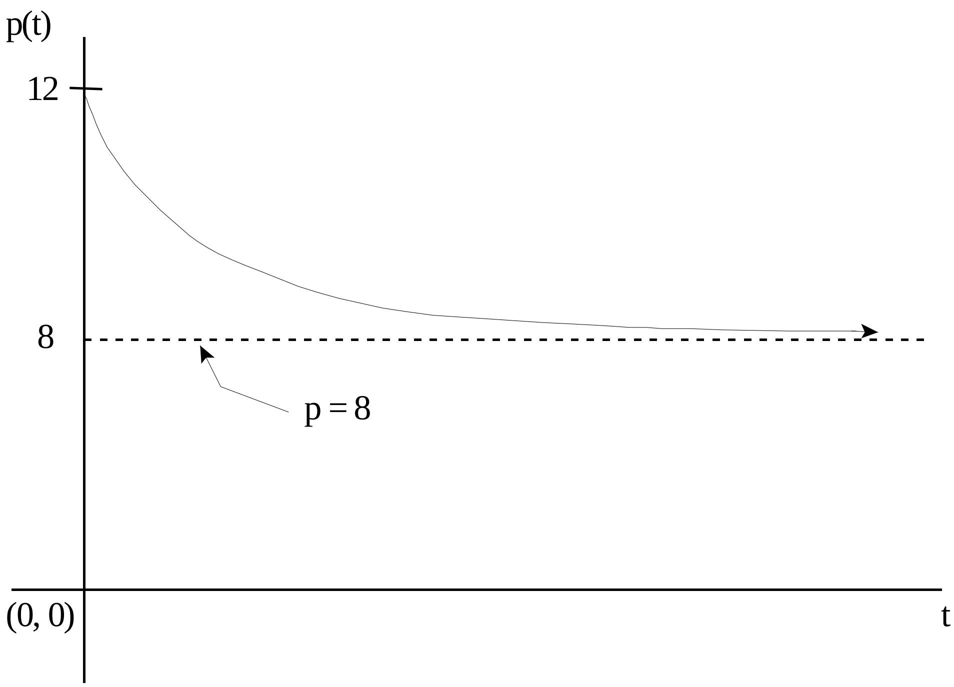
Hence, the graph of $p(t)$ is decreasing.

Next, let's consider the graph of $p(t)$ as $t \to \infty$.

$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \left(8 + 2te^{-2t} + 4e^{-2t}\right) = \lim_{t\to\infty} \left(8 + \frac{2t}{e^{2t}} + 4e^{-2t}\right) = \lim_{t\to\infty} \left(8 + \frac{2}{2e^{2t}} + 4e^{-2t}\right) = 8 + 0 + 0 = 8$

i.e., $\lim_{t\to\infty} p(t) = 8$

The graph of $y = p(t)$ is given below:



The market is **stable.** The equilibrium price is 8 units. The price will continue to decrease toward the equilibrium price $p_e = 8$.