MTH 1125 Test #4 - Solutions

SUMMER 2021

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Show CLEARLY how you arrive at your answers!

1. **Compute:** $\int \left(5x^4 + 8x^3 + 9x^2 + 4x + 6 + 6\sqrt{x}\right) dx =$

$$\int \left(5x^4 + 8x^3 + 9x^2 + 4x + 6 + 6\sqrt{x}\right) dx = \int \left(5x^4 + 8x^3 + 9x^2 + 4x + 6 + 6x^{\frac{1}{2}}\right) dx$$

$$= 5\left[\tfrac{x^5}{5}\right] + 8\left[\tfrac{x^4}{4}\right] + 9\left[\tfrac{x^3}{3}\right] + 4\left[\tfrac{x^2}{2}\right] + 6x + 6\left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right] + C = x^5 + 2x^4 + 3x^3 + 2x^2 + 6x + 4x^{\frac{3}{2}} + C$$

i.e., $\int \left(5x^4 + 8x^3 + 9x^2 + 4x + 6 + 6\sqrt{x}\right) dx = x^5 + 2x^4 + 3x^3 + 2x^2 + 6x + 4x^{\frac{3}{2}} + C$

2. **Compute:** $\int \left(\cos(x) + \sec^2(x)\right) dx =$

$$\int \left(\cos(x) + \sec^2(x)\right) dx = \sin(x) + \tan(x) + C$$

i.e., $\int \left(\cos(x) + \sec^2(x)\right) dx = \sin(x) + \tan(x) + C$

3. **Compute:** $\int_1^2 \left(4x^3 - 6x^2 + 4x\right) dx =$

$$\int_1^2 \underbrace{\left(4x^3 - 6x^2 + 4x\right)}_{f(x)} dx = \underbrace{\left[4\frac{x^4}{4} - 6\frac{x^3}{3} + 4\frac{x^2}{2}\right]_1^2}_{F(x)} = \underbrace{\left[x^4 - 2x^3 + 2x^2\right]_1^2}_{F(x)} =$$

$$= \underbrace{\left[(2)^4 - 2(2)^3 + 2(2)^2\right]}_{F(2)} - \underbrace{\left[(1)^4 - 2(1)^3 + 2(1)^2\right]}_{F(1)} = 8 - 1 = 7$$

i.e., $\int_1^2 \left(4x^3 - 6x^2 + 4x\right) dx = 7$

4. **Compute:** $\frac{d}{dx}\left[\ln\left(4x^2+8x+5\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(4x^2+8x+5\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{4x^2+8x+5}}_{\frac{1}{g(x)}} \cdot \underbrace{(8x+8)}_{g'(x)} = \frac{8x+8}{4x^2+8x+5}$$

i.e., $\frac{d}{dx}\left[\ln\left(4x^2+8x+5\right)\right] = \frac{8x+8}{4x^2+8x+5}$

5. **Compute:** $\frac{d}{dx}\left[\ln\left[\sqrt{\left(\frac{x^2+x}{x+2}\right)}\right]\right] \underbrace{=}_{\text{re-write}} \frac{d}{dx}\left[\ln\left[\left(\frac{x^2+x}{x+2}\right)^{\frac{1}{2}}\right]\right]$

Remark: We can compute this derivative directly, in it's current form, but it would be much easier to use the *algebraic properties* of natural logarithms to simplify the expression first.

$$\frac{d}{dx}\left[\ln\left[\left(\frac{x^2+x}{x+2}\right)^{\frac{1}{2}}\right]\right] = \underbrace{\frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{x^2+x}{x+2}\right)\right]}_{\ln(a^n)=n\ln(a)} = \underbrace{\frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(x^2+x\right)-\ln\left(x+2\right)\right)\right]}_{\ln\left(\frac{a}{b}\right)=\ln(a)-\ln(b)} = \frac{1}{2}\frac{d}{dx}\left[\ln\left(x^2+x\right)-\ln\left(x+\right.\right.$$

NOW we're ready to compute the derivative!

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2+x}{x+2}}\right)\right] = \frac{1}{2}\frac{d}{dx}\left[\ln\left(x^2+x\right)-\ln\left(x+2\right)\right] = \frac{1}{2}\left[\frac{1}{x^2+x}\left(2x+1\right)-\frac{1}{x+2}\right] = \frac{2x+1}{2x^2+2x} - \frac{1}{2x+4}$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2+x}{x+2}}\right)\right] = \frac{1}{2}\left[\frac{2x+1}{x^2+x}-\frac{1}{x+2}\right] = \frac{2x+1}{2x^2+2x} - \frac{1}{2x+4}$

6. **Compute:** $\int \frac{4x^2+2x}{4x^3+3x^2+5}dx =$

$$\int \frac{4x^2+2x}{4x^3+3x^2+5}dx \underbrace{=}_{\text{re-write}} \int \frac{1}{4x^3+3x^2+5}\left(4x^2+2x\right)dx$$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{4x^3+3x^2+5}$ is the same as $\left(4x^3+3x^2+5\right)^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = 4x^3+3x^2+5$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(4x^3+3x^2+5\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(4x^2+2x\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = 4x^3+3x^2+5$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 4x^3+3x^2+5 \\ \Rightarrow \frac{du}{dx} &= 12x^2+6x \\ \Rightarrow du &= \left(12x^2+6x\right)dx \\ \Rightarrow \frac{1}{3}du &= \left(4x^2+2x\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\underbrace{\int \frac{1}{4x^3+3x^2+5}}_{\frac{1}{u}} \underbrace{\left(4x^2+2x\right)dx}_{\frac{1}{3}du} = \int \frac{1}{u}\frac{1}{3}du = \frac{1}{3}\int \frac{1}{u}du$$

4. Integrate (in terms of u).

   $\frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| + C$

5. Re-express in terms of the original variable, x.

$$\int \tfrac{4x^2+2x}{4x^3+3x^2+5}dx = \underbrace{\frac{1}{3}\ln\left|4x^3+3x^2+5\right|+C}_{\frac{1}{3}\ln|u|+C}$$

$$\boxed{\text{i.e., } \int \tfrac{4x^2+2x}{4x^3+3x^2+5}dx = \tfrac{1}{3}\ln\left|4x^3+3x^2+5\right|+C}$$

7. **Compute:** $\int \cos\left(3x^2+5\right) x\,dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\cos\left(3x^2+5\right)$

   $\nearrow$ $\nwarrow$

   outer inner

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(3x^2+5\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(3x^2+5\right)}_{\text{function}} - - - - \rightarrow \underbrace{x}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(3x^2+5\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 3x^2+5 \\ \Rightarrow \frac{du}{dx} &= 6x \\ \Rightarrow du &= 6x\,dx \\ \Rightarrow \frac{1}{6}du &= x\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\cos\left(3x^2+5\right)}_{\cos(u)} \underbrace{x\,dx}_{\frac{1}{6}du} = \int \cos(u)\ \frac{1}{6}du = \frac{1}{6}\int \cos(u)\,du$$

4. Integrate (in terms of u).

$$\frac{1}{6}\int \cos(u)\,du = \frac{1}{6}\sin(u) + C$$

5. Re-express in terms of the original variable x.

$$\int \cos\left(3x^2+5\right) x\,dx = \underbrace{\frac{1}{6}\sin\left(3x^2+5\right)+C}_{\frac{1}{6}\sin(u)+C}$$

$$\text{i.e., } \int \cos\left(3x^2+5\right) x\,dx = \frac{1}{6}\sin\left(3x^2+5\right)+C$$

8. **Compute:** $\int \left(5x^2+5x+3\right)^4 (2x+1)\,dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(5x^2+5x+3\right)^4$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(5x^2+5x+3\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(5x^2+5x+3\right)}_{\text{function}} - - - - \rightarrow \underbrace{(2x+1)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(5x^2+5x+3\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 5x^2+5x+3 \\ \Rightarrow \frac{du}{dx} &= 10x+5 \\ \Rightarrow du &= (10x+5)\,dx \\ \Rightarrow \frac{1}{5}du &= (2x+1)\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(5x^2+5x+3\right)^4}_{u^9}\underbrace{(2x+1)\,dx}_{\frac{1}{5}du} = \int u^4 \tfrac{1}{5}du = \tfrac{1}{5}\int u^4 du$$

4. Integrate (in terms of u).

$$\tfrac{1}{5}\int u^4 du = \tfrac{1}{5}\left[\tfrac{u^5}{5}\right] + C = \tfrac{1}{25}u^5 + C$$

5. Re-express in terms of the original variable, x.

$$\int \left(5x^2+5x+3\right)^4 (2x+1)\,dx = \underbrace{\frac{1}{25}\left(5x^2+5x+3\right)^5 + C}_{\frac{1}{25}u^5+C}$$

i.e., $\int \left(5x^2+5x+3\right)^4 (2x+1)\,dx = \frac{1}{25}\left(5x^2+5x+3\right)^5 + C$

9. **Compute:** $\int_0^1 \left(x^2+1\right)^3 x\ dx =$ (The answer may not be a whole number)

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(x^2+1\right)^3$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(x^2+1\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(x^2+1\right)}_{\text{function}} - - - - \rightarrow \underbrace{x}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(x^2+1\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= x^2+1 \\ \Rightarrow \frac{du}{dx} &= 2x \\ \Rightarrow du &= 2x\ dx \\ \Rightarrow \frac{1}{2}du &= x\ dx \end{aligned}$$

   When $x = 0$, $u = x^2+1 = (0)^2+1 = 1$
   When $x = 1$, $u = x^2+1 = (1)^2+1 = 2$

3. Analyze in terms of u and du

$$\int_{x=0}^{x=1} \underbrace{\left(x^2+1\right)^3}_{u^3} \underbrace{x\ dx}_{\frac{1}{2}du} = \int_{u=1}^{u=2} u^3 \cdot \frac{1}{2}du = \frac{1}{2}\int_{u=1}^{u=2} u^3 du$$

   Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$\frac{1}{2}\int_{u=1}^{u=2} u^3 du = \frac{1}{2}\left[\frac{u^4}{4}\right]_{u=1}^{u=2} = \left[\frac{u^4}{8}\right]_{u=1}^{u=2} = \underbrace{\frac{(2)^4}{8}}_{F(2)} - \underbrace{\frac{(1)^4}{8}}_{F(1)} = \frac{16}{8} - \left(\frac{1}{8}\right) = \frac{15}{8}$$

$$\text{i.e., } \int_0^1 \left(x^2+1\right)^3 x\ dx = \frac{15}{8}$$