MTH 1126 - Test #1 - 9 am Class - Solutions

Spring 2024

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Show CLEARLY how you arrive at your answers

1. Compute: $\frac{d}{dx}\left[e^{\sin\left(6x^2\right)}\right] =$

$$\underbrace{\frac{d}{dx}\left[e^{\sin\left(6x^2\right)}\right]}_{\frac{d}{dx}[e^u]} = \underbrace{e^{\sin\left(6x^2\right)}}_{e^u} \cdot \underbrace{\frac{d}{dx}\left[\sin\left(6x^2\right)\right]}_{\frac{du}{dx}} = e^{\sin\left(6x^2\right)} \cdot \cos\left(6x^2\right) \cdot 12x$$

i.e., $\frac{d}{dx}\left[e^{\sin\left(6x^2\right)}\right] = e^{\sin\left(6x^2\right)} \cdot \cos\left(6x^2\right) \cdot 12x$

Or: $\frac{d}{dx}\left[e^{\sin\left(2x^2\right)}\right] = 12x\cos\left(6x^2\right)e^{\sin\left(6x^2\right)}$

2. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{2x^3+3x^2}{\tan(x)}}\right)\right] =$

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{2x^3+3x^2}{\tan(x)}}\right)\right] = \frac{d}{dx}\left[\ln\left(\left(\frac{2x^3+3x^2}{\tan(x)}\right)^{\frac{1}{2}}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{2x^3+3x^2}{\tan(x)}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(2x^3+3x^2\right) - \ln\left(\tan(x)\right)\right)\right]$$

$$= \frac{1}{2}\left(\underbrace{\frac{1}{2x^3+3x^2}}_{\frac{1}{u}} \cdot \underbrace{\left(6x^2+6x\right)}_{\frac{du}{dx}} - \underbrace{\frac{1}{\tan(x)}}_{\frac{1}{u}} \cdot \underbrace{\sec^2(x)}_{\frac{du}{dx}}\right) = \frac{1}{2}\left(\frac{6x^2+6x}{2x^3+3x^2} - \frac{\sec^2(x)}{\tan(x)}\right)$$

$$= \frac{1}{2}\left(\frac{6x+6}{2x^2+3x} - \cot(x)\sec^2(x)\right) = \frac{3x+3}{2x^2+3x} - \frac{1}{2}\cot(x)\sec^2(x)$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{2x^3+3x^2}{\tan(x)}}\right)\right] = \frac{3x+3}{2x^2+3x} - \frac{1}{2}\cot(x)\sec^2(x)$

(Alternative Solution Appears on the Following Page)

Alternative Solution:

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{2x^3+3x^2}{\tan(x)}}\right)\right] = \frac{d}{dx}\left[\underbrace{\ln\left[\left(\frac{2x^3+3x^2}{\tan(x)}\right)^{\frac{1}{2}}\right]}_{\ln(u)}\right] = \underbrace{\frac{1}{\left(\frac{2x^3+3x^2}{\tan(x)}\right)^{\frac{1}{2}}}}_{\frac{1}{u}} \cdot \underbrace{\left(\frac{d}{dx}\left[\left(\frac{2x^3+3x^2}{\tan(x)}\right)^{\frac{1}{2}}\right]\right)}_{\frac{du}{dx}}$$

$$= \left(\frac{2x^3+3x^2}{\tan(x)}\right)^{-\frac{1}{2}} \cdot \frac{1}{2}\left(\frac{2x^3+3x^2}{\tan(x)}\right)^{-\frac{1}{2}} \underbrace{\frac{(6x^2+6x)(\tan(x)) - (\sec^2(x))(2x^3+3x^2)}{(\tan(x))^2}}_{\text{Quotient Rule}}$$

$$= \left(\frac{\tan(x)}{2x^3+3x^2}\right)^{\frac{1}{2}} \cdot \frac{1}{2}\left(\frac{\tan(x)}{2x^3+3x^2}\right)^{\frac{1}{2}} \frac{\left(6x^2+6x\right)\tan(x) - \sec^2(x)\left(2x^3+3x^2\right)}{\tan^2(x)}$$

$$= \frac{1}{2}\left(\frac{\tan(x)}{2x^3+3x^2}\right) \frac{\left(6x^2+6x\right)\tan(x) - \sec^2(x)\left(2x^3+3x^2\right)}{\tan^2(x)} = \frac{1}{2}\left(\frac{1}{2x^3+3x^2}\right) \frac{\left(6x^2+6x\right)\tan(x) - \sec^2(x)\left(2x^3+3x^2\right)}{\tan(x)}$$

$$= \frac{\left(6x^2+6x\right)\tan(x) - \sec^2(x)\left(2x^3+3x^2\right)}{2(2x^3+3x^2)\tan(x)} = \frac{\left(6x^2+6x\right)\tan(x)}{2(2x^3+3x^2)\tan(x)} - \frac{\sec^2(x)\left(2x^3+3x^2\right)}{2(2x^3+3x^2)\tan(x)}$$

$$= \frac{3x+3}{2x^2+3x} - \frac{\sec^2(x)}{2\tan(x)} = \frac{3x+3}{2x^2+3x} - \frac{1}{2}\cot(x)\sec^2(x)$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{2x^3+3x^2}{\tan(x)}}\right)\right] = \frac{3x+3}{2x^2+3x} - \frac{1}{2}\cot(x)\sec^2(x)$

3. Compute: $\int e^{\left(5x^4+4x^3\right)}\left(5x^3+3x^2\right)\,dx =$

(a)

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes. $e^{\left(5x^4+4x^3\right)}$

   Let $u =$ the "inner function"

   i.e., $u = 5x^4 + 4x^3$

   b. Is there an "approximate function/derivative pair"?

   Yes. $\underbrace{\left(5x^4+4x^3\right)}_{\text{function}} \rightarrow \underbrace{\left(5x^3+3x^2\right)}_{\text{deriv}}$

   Let $u =$ the "function"

   i.e., $u = (5x^4 + 4x^3)$

2. Compute du

$$\begin{array}{rrcl} & u & = & 5x^4+4x^3 \\ \Rightarrow & \frac{du}{dx} & = & 20x^3+12x^2 \\ \Rightarrow & du & = & (20x^3+12x^2)\,dx \\ \Rightarrow & \frac{1}{4}du & = & (5x^3+3x^2)\,dx \end{array}$$

3. Analyze in terms of u and du.

$$\int \underbrace{e^{\left(5x^4+4x^3\right)}}_{e^u}\underbrace{\left(5x^3+3x^2\right)\,dx}_{\frac{1}{4}du} = \int e^u \tfrac{1}{4}du = \tfrac{1}{4}\int e^u\,du$$

4. Integrate in terms of u

$\frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + C$

5. Re-write in terms of x

$$\int e^{\left(5x^4+4x^3\right)}\left(5x^3+3x^2\right)\,dx = \underbrace{\frac{1}{4}e^{\left(5x^4+4x^3\right)} + C}_{\frac{1}{4}e^u+C}$$

$$\boxed{\text{i.e., } \int e^{\left(5x^4+4x^3\right)}\left(5x^3+3x^2\right)\,dx = \tfrac{1}{4}e^{\left(5x^4+4x^3\right)} + C}$$

4. Compute: $\int \frac{3x^5+4x^3}{(3x^6+6x^4)^4}\,dx = \int \frac{1}{(3x^6+6x^4)^4}\left(3x^5+4x^3\right)dx = \int \left(3x^6+6x^4\right)^{-4}\left(3x^5+4x^3\right)dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

      Yes. $\left(3x^6+6x^4\right)^{-4}$

      Let $u =$ the "inner function"

      Let $u = \left(3x^6+6x^4\right)$

   b. Is there an "approximate function/derivative pair"?

      Yes. $\underbrace{\left(3x^6+6x^4\right)}_{\text{function}} \rightarrow \underbrace{\left(3x^5+4x^3\right)}_{\text{deriv}}$

      Let $u =$ the "function"

      Let $u = \left(3x^6+6x^4\right)$

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & 3x^6+6x^4 \\ \Rightarrow & \frac{du}{dx} & = & 18x^5+24x^3 \\ \Rightarrow & du & = & \left(18x^5+24x^3\right)dx \\ \Rightarrow & \frac{1}{6}du & = & \left(3x^5+4x^3\right)dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\left(3x^6+6x^4\right)^{-4}}_{u^{-4}}\underbrace{\left(3x^5+4x^3\right)dx}_{\frac{1}{6}du} = \int u^{-4}\tfrac{1}{6}du = \tfrac{1}{6}\int u^{-4}\,du$$

4. Integrate in terms of u

$$\tfrac{1}{6}\int u^{-4}\,du = \tfrac{1}{6}\tfrac{u^{-3}}{-3} + C = -\tfrac{1}{18}u^{-3} + C$$

5. Re-write in terms of x

$$\int \frac{3x^5+4x^3}{(3x^6+6x^4)^4}\,dx = \underbrace{-\frac{1}{18}\left(3x^6+6x^4\right)^{-3} + C}_{-\frac{1}{3}u^{-3}+C}$$

$$\boxed{\text{i.e., } \int \frac{3x^5+4x^3}{(3x^6+6x^4)^4}\,dx = -\tfrac{1}{18}\left(3x^6+6x^4\right)^{-3} + C}$$

5. Compute: $\int \frac{x^5+x+1}{(x^6+3x^2+6x)}\,dx = \int \frac{1}{(x^6+3x^2+6x)}\,(x^5+x+1)\,dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes. $\frac{1}{(x^6+3x^2+6x)} = (x^6+3x^2+6x)^{-1}$

   Let $u = (x^6+3x^2+6x)$ i.e., "Let $u =$ the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

   Yes. $\underbrace{(x^6+3x^2+6x)}_{\text{function}} \to \underbrace{(x^5+x+1)}_{\text{deriv}}$

   Let $u = (x^6+3x^2+6x)$ i.e., "Let $u =$ 'the function'"

2. Compute du

$$\begin{array}{rrcl} & u & = & x^6+3x^2+6x \\ \Rightarrow & \frac{du}{dx} & = & 6x^5+6x+6 \\ \Rightarrow & du & = & (6x^5+6x+6)\,dx \\ \Rightarrow & \frac{1}{6}du & = & (x^5+x+1)\,dx \end{array}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\frac{1}{(x^6+3x^2+6x)}}_{\frac{1}{u}} \underbrace{(x^5+x+1)\,dx}_{\frac{1}{6}du} = \int \frac{1}{u}\frac{1}{6}du = \frac{1}{6}\int \frac{1}{u}\,du$$

4. Integrate in terms of u

$\frac{1}{6}\int \frac{1}{u}\,du = \frac{1}{6}\ln|u| + C$

5. Re-write in terms of x

$$\int \frac{x^5+x+1}{(x^6+3x^2+6x)}\,dx = \underbrace{\frac{1}{6}\ln\left|x^6+3x^2+6x\right| + C}_{\frac{1}{6}\ln|u|+C}$$

$$\text{i.e., } \int \frac{x^5+x+1}{(x^6+3x^2+6x)}\,dx = \frac{1}{6}\ln|x^6+3x^2+6x| + C$$

6. Compute: $\frac{d}{dx}\left[\arctan\left(\sin\left(x\right)\right)\right] =$

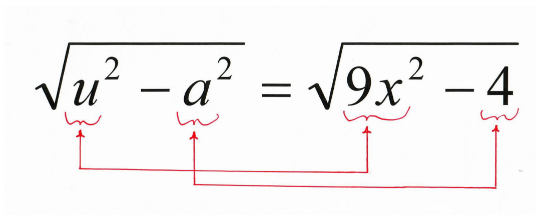
$$\underbrace{\frac{d}{dx}\left[\arctan\left(\sin\left(x\right)\right)\right]}_{\frac{d}{dx}[\arctan(u)]} = \underbrace{\frac{1}{1+\left(\sin\left(x\right)\right)^2}}_{\frac{1}{1+u^2}} \cdot \underbrace{\cos\left(x\right)}_{\frac{du}{dx}} = \frac{\cos(x)}{1+\sin^2(x)}$$

i.e., $\frac{d}{dx}\left[\arctan\left(\sin\left(x\right)\right)\right] = \frac{\cos(x)}{1+\sin^2(x)}$

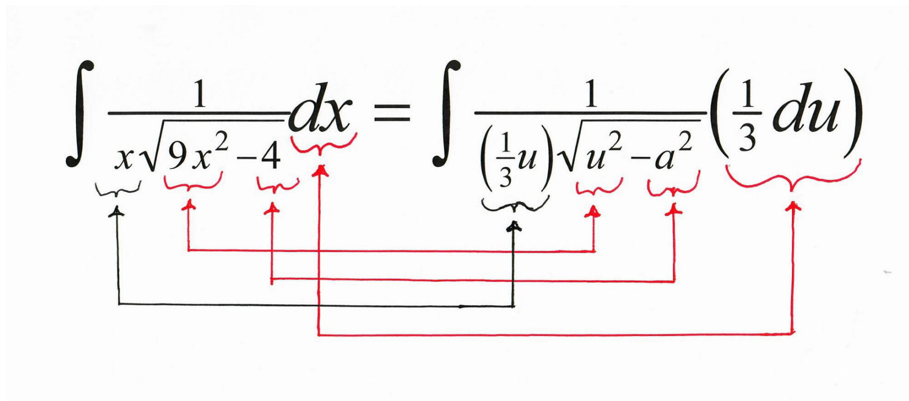
7. Compute: $\int \frac{1}{x\sqrt{9x^2-4}}\,dx =$

This appears to fit the form: $\int \frac{1}{u\sqrt{u^2-a^2}}du$

If our conjecture is correct, then $\sqrt{u^2-a^2} = \sqrt{9x^2-4}$



$$\begin{aligned}
\Rightarrow\quad & a^2 = 4 \\
& a = 2 \\
\Rightarrow\quad & u^2 = 9x^2 \\
& u = 3x \\
& \tfrac{1}{3}u = x \\
\Rightarrow\quad & \tfrac{du}{dx} = 3 \\
& du = 3dx \\
& \tfrac{1}{3}du = dx
\end{aligned}$$



3. Analyze in terms of u and du.

$$\int \frac{1}{x\sqrt{9x^2-4}}\,dx = \int \frac{1}{x\sqrt{(3x)^2-2^2}}dx = \int \frac{1}{\left(\frac{1}{3}u\right)\sqrt{u^2-a^2}}\left(\tfrac{1}{3}du\right) = \int \frac{1}{u\sqrt{u^2-a^2}}du$$

4. Integrate

$$\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$$

5. Re-express in terms of x

$$\int \frac{1}{x\sqrt{9x^2-4}}\,dx = \underbrace{\frac{1}{2}\operatorname{arcsec}\left(\frac{|3x|}{2}\right) + C}_{\frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)+C}$$

$$\boxed{\int \frac{1}{x\sqrt{9x^2-4}}\,dx = \frac{1}{2}\operatorname{arcsec}\left(\frac{|3x|}{2}\right) + C}$$

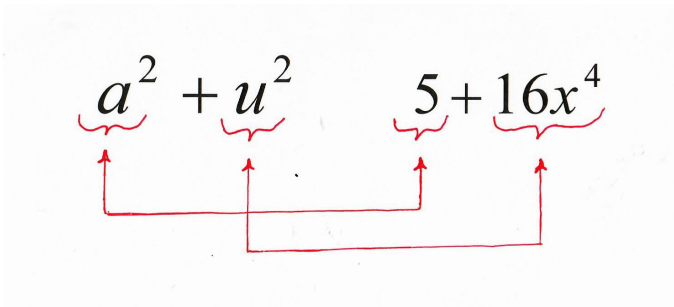
8. Compute: $\frac{d}{dx}\left[\cos^{-1}\left(e^{x}\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\cos^{-1}\left(e^{x}\right)\right]}_{\frac{d}{dx}[\cos^{-1}(u)]} = \underbrace{-\frac{1}{\sqrt{1-\left(e^{x}\right)^{2}}}}_{-\frac{1}{\sqrt{1-u^2}}} \cdot \underbrace{e^{x}}_{\frac{du}{dx}} = -\frac{1}{\sqrt{1-e^{2x}}} \cdot e^{x} = -\frac{e^{x}}{\sqrt{1-e^{2x}}}$$

i.e., $\frac{d}{dx}\left[\cos^{-1}\left(e^{x}\right)\right] = -\frac{e^{x}}{\sqrt{1-e^{2x}}}$

9. Compute: $\int \frac{x}{5+16x^4}\,dx =$

$\int \frac{x}{5+16x^4}\,dx = \int \frac{1}{5+16x^4}x\,dx$ (We'll try to make this fit the form: $\int \frac{1}{a^2+u^2}du$)



$$\boxed{\begin{array}{ll} \Rightarrow & a^2 = 5 \\ & a = \sqrt{5} \\ \Rightarrow & u^2 = 16x^4 \\ & u = 4x^2 \\ \Rightarrow & \frac{du}{dx} = 8x \\ & du = 8xdx \\ & \frac{1}{8}du = xdx \end{array}}$$

Thus, we have:

$$\int \frac{x}{5+16x^4}\,dx = \int \frac{1}{5+16x^4}x\,dx = \int \underbrace{\frac{1}{\left(\sqrt{5}\right)^2 + (4x^2)^2}}_{\frac{1}{a^2+u^2}} \underbrace{xdx}_{\frac{1}{8}du} = \int \frac{1}{a^2+u^2}\frac{1}{8}du = \frac{1}{8}\int \frac{1}{a^2+u^2}du$$

$$= \frac{1}{8}\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C = \frac{1}{8\sqrt{5}}\tan^{-1}\left(\frac{4x^2}{\sqrt{5}}\right) + C$$

$$\boxed{\text{i.e., } \int \frac{x}{5+16x^4}\,dx = \frac{1}{8\sqrt{5}}\tan^{-1}\left(\frac{4x^2}{\sqrt{5}}\right) + C}$$

10. $z = \sec(\arcsin(2x))$ Re-write this equation as an equivalent algebraic equation.

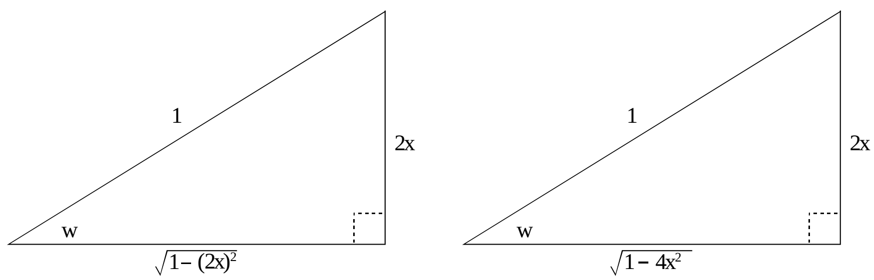
To make the equation a little easier to work with, we'll let $w = \arcsin(2x)$

Then "w is the angle whose sine is $2x$."

i.e., $\sin(w) = 2x$

We'll draw a right triangle that depicts this relationship.

i.e., $\sin(w) = 2x = \frac{\text{opp}}{\text{hyp}} = \frac{2x}{1}$



$adj^2 = hyp^2 - opp^2 = 1^2 - (2x)^2 = 1 - 4x^2$
i.e., $adj^2 = 1 - 4x^2$
$\Rightarrow adj = \sqrt{1 - 4x^2}$
We want $z = \sec(\arcsin(2x))$

But since $w = \arcsin(2x)$,

$\Rightarrow z = \sec(w)$

$\Rightarrow z = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\sqrt{1-4x^2}}$

i.e., $z = \frac{1}{\sqrt{1-4x^2}}$

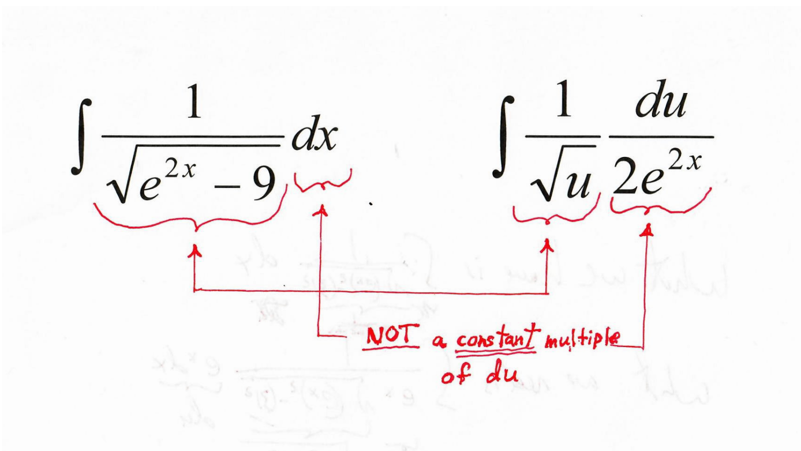
Extra: Wow! 10 points (All or nothing)

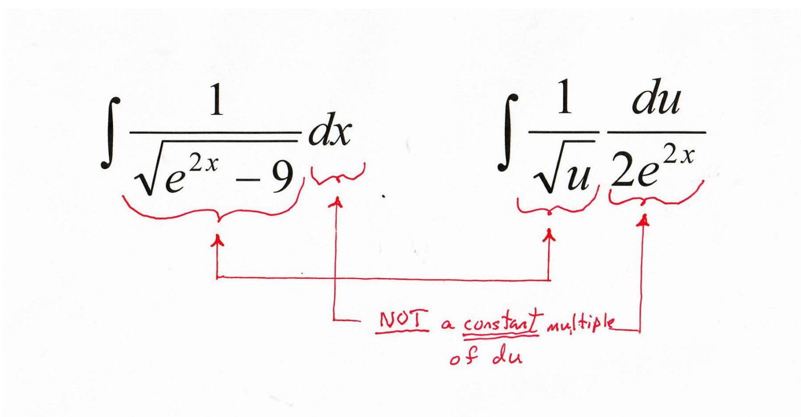
Compute: $\int \frac{1}{\sqrt{e^{2x}-9}}\,dx =$

Q: Does this fit the form $\int \frac{1}{\sqrt{u}} du$??

If it does, then:

$$\begin{array}{|rcl|} u & = & e^{2x}-9 \\ \frac{du}{dx} & = & 2e^{2x} \\ du & = & 2e^{2x}dx \\ \frac{du}{2e^{2x}} & = & dx \end{array}$$





Since dx is not a constant multiple of du, we can't make this integral fit the form: $\int \frac{1}{\sqrt{u}} du$.

Q: Can we make this fit the form: $\int \frac{1}{u\sqrt{u^2-a^2}} du$???

At first, this doesn't seem like it will work.

But let's pursue this course and "see it through."

$$\int \frac{1}{\sqrt{\underbrace{e^{2x}}_{u^2} - \underbrace{9}_{a^2}}}\,dx$$

This gives us:

u^2	$=$	e^{2x}
u	$=$	e^x
$\frac{du}{dx}$	$=$	e^x
du	$=$	$e^x dx$

What we HAVE is $\int \frac{1}{\underbrace{\sqrt{(e^x)^2 - 3^2}}_{\sqrt{u^2-a^2}}} dx$

What we NEED is $\int \frac{1}{\underbrace{e^x}_{u}\underbrace{\sqrt{(e^x)^2 - 3^2}}_{\sqrt{u^2-a^2}}} \underbrace{e^x dx}_{du}$

a^2	$=$	9
a	$=$	3
u^2	$=$	$(e^x)^2$
u	$=$	e^x
$\frac{du}{dx}$	$=$	e^x
du	$=$	$e^x dx$

We CAN rewrite the integrand in exactly this form!

$$\int \frac{1}{\sqrt{e^{2x}-9}}\,dx = \int \frac{1}{\sqrt{(e^x)^2-3^2}}\,dx = \int \underbrace{\frac{e^x}{e^x}}_{=1} \cdot \frac{1}{\sqrt{(e^x)^2-3^2}}\,dx = \int \frac{1}{\underbrace{e^x}_{u}\underbrace{\sqrt{(e^x)^2 - 3^2}}_{\sqrt{u^2-a^2}}} \underbrace{e^x dx}_{du}$$

$$= \int \frac{1}{u\sqrt{u^2-a^2}} du = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C = \frac{1}{3}\sec^{-1}\left(\frac{|e^x|}{3}\right) + C$$

i.e., $\int \frac{1}{\sqrt{e^{2x}-9}}\,dx = \frac{1}{3}\sec^{-1}\left(\frac{|e^x|}{3}\right) + C$