MTH 1126 Test #3 - Solutions

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Show CLEARLY how you arrive at your answers.

1. $\int xe^x dx =$

   When doing integration by parts, one rule of thumb is to let u be a portion of the integrand whose derivative is simpler than itself. By this criterion, $u = x$ qualifies, because the derivative of x is a constant - simpler than x itself. Another rule of thumb is to let dv be the largest (or most complicated) portion of the integrand that we can integrate. By this criterion, we have $dv = e^x dx$.

   $\Rightarrow u = x$ and $dv = e^x dx$.

   Consequently, we let

$u = x$		$dv = e^x dx$
$\frac{du}{dx} = 1$		$\int dv = \int e^x dx$
$du = dx$		$v = e^x$

   Thus we have:

   $$\int \underbrace{x}_{u} \underbrace{e^x dx}_{dv} = \int u\,dv = uv - \int v du = xe^x - \int e^x dx = xe^x - e^x + C$$

   i.e., $\int xe^x dx = xe^x - e^x + C$

2. $\lim_{x\to 0} \frac{\cos(x) - e^{-x}}{x} =$

   $\lim_{x\to 0} \frac{\cos(x) - e^{-x}}{x} \sim \frac{0}{0}$ so we can use L'Hopital's Rule

   $$\Rightarrow \underbrace{\lim_{x\to 0} \frac{\cos(x) - e^{-x}}{x} = \lim_{x\to 0} \frac{-\sin(x) + e^{-x}}{1}}_{\text{By L'Hopital's Rule}} = \frac{0+1}{1} = 1$$

   i.e., $\lim_{x\to 0} \frac{\cos(x) - e^{-x}}{x} = 1$

3. $\int \frac{12}{x^2-4} dx =$

Factor the denominator: $\frac{12}{(x+2)(x-2)}$

Observe:

$\frac{12}{(x+2)(x-2)} = \frac{C_1}{x+2} + \frac{C_2}{x-2}$

Multiply both sides by the denominator

$\Rightarrow 12 = \frac{C_1}{x+2}(x+2)(x-2) + \frac{C_2}{x-2}(x+2)(x-2)$

$\Rightarrow 12 = C_1(x-2) + C_2(x+2)$

Solve for the constants, by plugging in "strategic values" of x.

$\boxed{x=2}$ We have: $12 = C_2(4)$

$\boxed{\Rightarrow C_2 = 3}$

$\boxed{x=-2}$ We have: $12 = C_1(-4)$

$\boxed{\Rightarrow C_1 = -3}$

Therefore:

$\int \frac{12}{x^2-4} dx = \int \left(\frac{-3}{x+2} + \frac{3}{x-2}\right) dx = -3 \int \frac{1}{x+2} dx + 3 \int \frac{1}{x-2} dx = -3 \ln|x+2| + 3 \ln|x-2| + C$

$$\boxed{\text{i.e., } \int \frac{12}{x^2-4} dx = -3 \ln|x+2| + 3 \ln|x-2| + C}$$

4. $\int \sin^3(x)\sqrt{\cos(x)}dx =$

(sine to an odd power) Pull out a factor of $\sin(x)$ to serve as the "future du".

$= \int \sin^2(x)\cos^{\frac{1}{2}}(x)\underbrace{\sin(x)\,dx}_{\text{"future du"}}$

(Clearly, we intend to let $u = \cos(x)$)

Convert the rest of the sines to cosines using the identity $\sin^2(x) = 1 - \cos^2(x)$.

$= \int (1 - \cos^2(x))\cos^{\frac{1}{2}}(x)\sin(x)\,dx.$

Let $u = \cos(x)$

$\Rightarrow du = -\sin(x)\,dx$

$\Rightarrow -du = \sin(x)\,dx$

Continuing, we have:

$$\int \underbrace{(1-\cos^2(x))}_{1-u^2}\underbrace{\cos^{\frac{1}{2}}(x)}_{u^{\frac{1}{2}}}\underbrace{\sin(x)\,dx}_{-du} = \int (1-u^2)\,u^{\frac{1}{2}}\,(-du) = \int (u^2-1)\,u^{\frac{1}{2}}\,du$$

$$= \int \left(u^{\frac{5}{2}} - u^{\frac{1}{2}}\right)du = \frac{2}{7}u^{\frac{7}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{7}\cos^{\frac{7}{2}}(x) - \frac{2}{3}\cos^{\frac{3}{2}}(x) + C$$

i.e., $\int \sin^3(x)\sqrt{\cos(x)}dx = \frac{2}{7}\cos^{\frac{7}{2}}(x) - \frac{2}{3}\cos^{\frac{3}{2}}(x) + C$

5. $\int \frac{1}{x^2\sqrt{16-4x^2}}dx =$

Use trig substitution to get rid of the radical. We want to replace $16 - 4x^2$ with something of the form $a^2 - a^2\sin^2(\theta)$.

$\sqrt{16-4x^2} \Rightarrow\Rightarrow\Rightarrow \sqrt{a^2 - a^2\sin^2(\theta)}$

Matching corresponding parts, we have:

$a^2 = 16$

$\Rightarrow a = 4$

Also:

$4x^2 = a^2\sin^2(\theta)$

$\Rightarrow x^2 = \frac{a^2}{4}\sin^2(\theta)$

$\Rightarrow x = \frac{a}{2}\sin(\theta)$

$\Rightarrow dx = \frac{a}{2}\cos(\theta)\,d\theta$

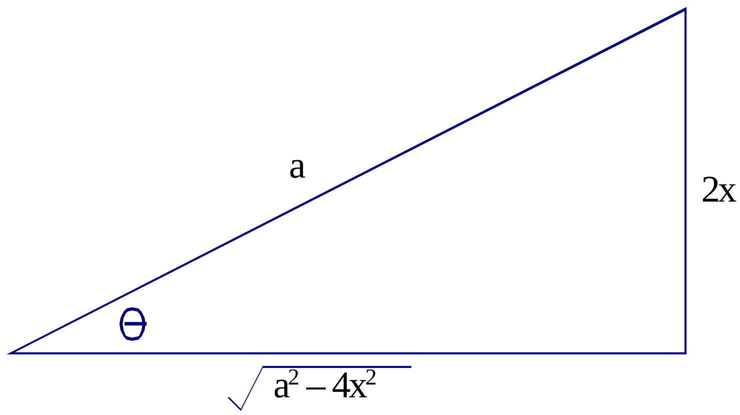
Thus, we have:

$$\int \frac{1}{x^2\sqrt{16-4x^2}}dx = \int \frac{1}{\frac{a^2}{4}\sin^2(\theta)\sqrt{a^2-a^2\sin^2(\theta)}}\frac{a}{2}\cos(\theta)\,d\theta = \int \frac{1}{\frac{a^2}{4}\sin^2(\theta)\sqrt{a^2\cos^2(\theta)}}\frac{a}{2}\cos(\theta)\,d\theta$$

$$= \int \frac{1}{\frac{a^2}{4}\sin^2(\theta)a\cos(\theta)}\frac{a}{2}\cos(\theta)\,d\theta = \int \frac{1}{\frac{a^2}{4}\sin^2(\theta)}\frac{1}{2}d\theta = \frac{2}{a^2}\int \csc^2(\theta)\,d\theta$$

$$= -\frac{2}{a^2}\cot(\theta) + C$$

Observe: $x = \frac{a}{2}\sin(\theta) \Rightarrow \sin(\theta) = \frac{2x}{a} = \frac{\text{opp}}{\text{hyp}}$. This yields the triangle below:



Continuing where we left off:

$$\int \frac{1}{x^2\sqrt{16-4x^2}}dx = -\frac{2}{a^2}\cot(\theta) + C = -\frac{2}{16}\frac{\sqrt{a^2-4x^2}}{2x} + C = -\frac{\sqrt{16-4x^2}}{16x} + C$$

i.e., $\int \frac{1}{x^2\sqrt{16-4x^2}}dx = -\frac{\sqrt{16-4x^2}}{16x} + C$

6. $\lim_{x\to\infty} \frac{\ln(x)}{e^x} =$

$\lim_{x\to\infty} \frac{\ln(x)}{e^x} \sim \frac{\infty}{\infty}$ so we can use L'Hopital's Rule

$$\Rightarrow \underbrace{\lim_{x\to\infty} \frac{\ln(x)}{e^x} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{e^x}}_{\text{By L'Hopital's Rule}} = \lim_{x\to\infty} \frac{1}{xe^x} = 0$$

i.e., $\lim_{x\to\infty} \frac{\ln(x)}{e^x} = 0$

7. $\int \sin^2(x)\,dx =$

(sine and cosine raised to even powers) Reduce the powers of sine and cosine by using the double angle formulas:

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \quad \text{and} \quad \cos^2(x) = \frac{1+\cos(2x)}{2}$$

Continuing:

$$\int \sin^2(x)\,dx = \int \left(\frac{1-\cos(2x)}{2}\right) dx = \frac{1}{2}\int (1-\cos(2x))\,dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin(2x)\right) + C = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

i.e., $\int \sin^2(x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$

WOW! Extra (10 pts - all or nothing)

$\int e^x \sin(x)\,dx =$

Here, both factors of the integrand have derivatives that are cyclic, so it won't matter which we choose to be u and which we choose to be dv. Also, since both factors of the integrand have derivatives that are cyclic, we expect that we will integrate by parts more than once, and eventually end up with the original integral on the right side of the equation. We will then solve for the integral algebraically. Let

u	$=$	e^x	dv	$=$	$\sin(x)\,dx$
$\frac{du}{dx}$	$=$	e^x	$\int dv$	$=$	$\int \sin(x)\,dx$
du	$=$	$e^x dx$	v	$=$	$-\cos(x)$

Thus, we have:

$$\int \underbrace{e^x}_{u} \underbrace{\sin(x)\,dx}_{dv} = \int u\,dv = uv - \int v\,du = e^x(-\cos(x)) - \int (-\cos(x))\,e^x dx$$

$$= -e^x \cos(x) + \int e^x \cos(x)\,dx$$

At this point, we repeat the procedure, being sure not to "switch" our choices of u and dv (i.e. we continue to let u be e^x and we let dv be the trig function). This yields:

u	$=$	e^x	dv	$=$	$\cos(x)\,dx$
$\frac{du}{dx}$	$=$	e^x	$\int dv$	$=$	$\int \cos(x)\,dx$
du	$=$	$e^x dx$	v	$=$	$\sin(x)$

Continuing, we have:

$$-e^x \cos(x) + \int \underbrace{e^x}_{u} \underbrace{\cos(x)\,dx}_{dv} = -e^x \cos(x) + \int u\,dv = -e^x \cos(x) + uv - \int v\,du$$

$$= -e^x \cos(x) + e^x \sin(x) - \int \sin(x)\,e^x dx = -e^x \cos(x) + e^x \sin(x) - \underbrace{\int e^x \sin(x)\,dx}_{\text{what we started with}}$$

We sum up what we've established so far:

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\,dx$$

We can solve for $\int e^x \sin(x)\,dx$ algebraically.

$$\Rightarrow 2\int e^x \sin(x)\,dx = -e^x \cos(x) + e^x \sin(x)$$

$$\Rightarrow \int e^x \sin(x)\,dx = -\tfrac{1}{2}e^x \cos(x) + \tfrac{1}{2}e^x \sin(x) + C$$

i.e., $\int e^x \sin(x)\,dx = -\frac{1}{2}e^x \cos(x) + \frac{1}{2}e^x \sin(x) + C$