Solutions.

1. $\int \frac{3t^2}{(2t^3+1)^{\frac{1}{2}}} dt = \int (2t^3+1)^{-\frac{1}{2}} 3t^2 dt$

    1. U-sub appropriate?

        1. Composite function? Yes!

        $(2t^3+1)^{-\frac{1}{2}}$
        Let $u = 2t^3+1$

        1. Approximate function/derivative pair? Yes!

        $\underbrace{2t^3+1}_{\text{function}} \rightarrow \underbrace{3t^2}_{\text{deriv}}$
        Let $u = 2t^3+1$

    2. Compute du
    $u = 2t^3+1$
    $\frac{du}{dt} = 6t^2$
    $du = 6t^2 dt$
    $\frac{1}{2} du = 3t^2 dt$

    3. Analyze in terms of u and du
    $\int \underbrace{\left(2t^3+1\right)^{-\frac{1}{2}}}_{u^{-\frac{1}{2}}} \underbrace{3t^2 dt}_{\frac{1}{2}du} = \int u^{-\frac{1}{2}} \frac{1}{2} du = \frac{1}{2} \int u^{-\frac{1}{2}} du$

    4. Integrate:
    $= \frac{1}{2} \int u^{-\frac{1}{2}} du = \frac{1}{2} \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right] + C = u^{\frac{1}{2}} + C$

    5. Restate in terms of t
    $= (2t^3+1)^{\frac{1}{2}} + C$

2. $\frac{d}{dx}\left[\underbrace{\cos}_{\text{outer}} \underbrace{\left(4x^2+3x+2\right)}_{\text{inner}}\right] = \underbrace{\underbrace{\left[-\sin\left(4x^2+3x+2\right)\right]}_{\text{deriv of outer; evulated at inner}} \underbrace{(8x+3)}_{\text{deriv of inner}}}_{\text{By Chain Rule}}$

3. $\frac{d}{dx}\left[\underbrace{\sin}_{\text{outer}} \underbrace{\cos(x)}_{\text{inner}}\right] = \underbrace{[\cos(\cos x)]}_{\text{deriv of outer; evaluated at inner}} \underbrace{(-\sin x)}_{\text{deriv of inner}} = -\cos(\cos x)\sin x$

For problems 4 and 5, suppose $P(x)$ has the property that $P'(x) = \frac{1}{x}$

4. Compute: $\frac{d}{dx}[P(\sin(x))] = \frac{1}{\underbrace{\sin x}_{\text{deriv of outer evaluated at inner}}} \cdot \underbrace{\cos x}_{\text{deriv of inner}} = \frac{\cos(x)}{\sin(x)} = \cot(x)$

5. Compute: $\frac{d}{dx}[P(4x^3+7x^2)] = \frac{1}{\underbrace{4x^3+7x^2}_{\text{deriv of outer evaluated at inner}}} \cdot \underbrace{\left(12x^2+14x\right)}_{\text{dreiv of inner}} = \frac{12x^2+14x}{4x^3+7x^2}$

6. $\int \frac{4x}{(9x^2+3)^2}dx = \int (9x^2+3)^{-2}\, 4xdx$

   1. U-sub appropriate?

      1. Composite function? Yes!
         $(9x^2+3)^{-2}$
         Let $u = 9x^2+3$

      2. Approximate funct/deriv pair? Yes!
         $\underbrace{\left(9x^2+3\right)}_{\text{function}} \rightarrow \underbrace{4x}_{\text{deriv.}}$
         Let $u = 9x^2+3$

   2. Compute du
      $u = 9x^2+3$
      $du = 18xdx$
      $\frac{1}{18}du = xdx$
      $\frac{4}{18}du = 4xdx$
      $\frac{2}{9}du = 4xdx$

   3. Analyze in terms of u and du.
      $\int \underbrace{\left(9x^2+3\right)^{-2}}_{u^{-2}}\underbrace{4xdx}_{\frac{2}{9}du} = \int u^{-2}\frac{2}{9}du = \frac{2}{9}\int u^{-2}du$

   4. Integrate:
      $\frac{2}{9}\int u^{-2}du = \frac{2}{9}\left[\frac{u^{-1}}{-1}\right] + C = -\frac{2}{9}u^{-1} + C$

   5. Restate in terms of x
      $= -\frac{2}{9}\left(9x^2+3\right)^{-1} + C$

7. $f(x) = 3\sin(x) - 4\cos x$; $f'(x) = 3[\cos(x)] - 4[-\sin(x)] = 3\cos(x) + 4\sin(x)$

8. $\frac{d}{dx}\left[\cos^5(x)\right] = \underbrace{\frac{d}{dx}\left[(\cos(x))^5\right]}_{[g(x)]^n} = \underbrace{5\left[\cos(x)\right]^4}_{n[g(x)]^{n-1}} \cdot \underbrace{(-\sin x)}_{g'(x)} = -5\cos^4(x)\sin x$

9. $\frac{d}{dx}\left[\sin^2\left(6x^2+3x\right)\right] = \frac{d}{dx}\underbrace{\left[\left(\sin\left(6x^2+3x\right)\right)^2\right]}_{[g(x)]^n}$

$$=\underbrace{\underbrace{2\left[\sin\left(6x^2+3x\right)\right]}_{n[g(x)]^{n-1}} \cdot \underbrace{\frac{d}{dx}\left[\sin\left(6x^2+3x\right)\right]}_{\frac{d}{dx}[g(x)]}}_{\text{By Chain Rule}}$$

$$= 2\left[\sin\left(6x^2+3x\right)\right]\underbrace{\underbrace{\left[\cos\left(6x^2+3x\right)\right]}_{\substack{\text{deriv of outer}\\ \text{evaluated at inner}}}\underbrace{[12x+3]}_{\text{deriv of inner}}}_{\text{By Chain Rule}}$$

10. $\int (1+\cos x)^{\frac{3}{2}} \sin x \; dx$

   1. U-sub appropriate?

1. Composite function? Yes!
   $(1+\cos(x))^{\frac{3}{2}}$
   let $u = 1+\cos(x)$
2. Approx. funct/deriv Pair? Yes!
   $\underbrace{(1+\cos(x))}_{\text{function}} \rightarrow \underbrace{\sin(x)}_{\text{deriv.}}$
   let $u = 1+\cos(x)$

2. Compute du
   $u = 1+\cos x$
   $\frac{du}{dx} = -\sin(x)$
   $du = -\sin(x)\,dx$
   $-du = \sin(x)\,dx$
3. Analyze in terms of u and du
   $\int \underbrace{(1+\cos(x))^{\frac{3}{2}}}_{u^{\frac{3}{2}}} \underbrace{\sin(x)\,dx}_{-du} = \int u^{\frac{3}{2}}(-du) = -\int u^{\frac{3}{2}} du$
4. Integrate
   $-\int u^{\frac{3}{2}} du = -\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + C = -\frac{2}{5}u^{\frac{5}{2}} + C$
5. Restate in terms of x
   $= -\frac{2}{5}(1+\cos(x))^{\frac{5}{2}} + c$

11. $\int \frac{\sin x}{\sqrt{\cos x}} dx = \int (\cos(x))^{-\frac{1}{2}} \sin(x)\,dx$
    1. U-sub appropriate?
       1. Composite Function? Yes!
          $(\cos(x))^{-\frac{1}{2}}$
          Let $u = \cos(x)$
       2. Approx funct/deriv pair? Yes!
          $\underbrace{(\cos(x))}_{\text{function}} \rightarrow \underbrace{(\sin(x))}_{\text{deriv}}$
          Let $u = \cos(x)$
    2. Compute du
       $u = \cos(x)$
       $\frac{du}{dx} = -\sin(x)$
       $du = -\sin(x)\,dx$
       $-du = \sin(x)\,dx$
    3. Analyze in terms of u and du.
       $\int \underbrace{(\cos(x))^{-\frac{1}{2}}}_{u^{-\frac{1}{2}}} \underbrace{\sin(x)\,dx}_{-du} = \int u^{-\frac{1}{2}}(-du) = -\int u^{-\frac{1}{2}} du$
    4. Integrate
       $-\int u^{-\frac{1}{2}} du = -\left[\frac{u^{-\frac{1}{2}}}{\frac{1}{2}}\right] + C = -2u^{\frac{1}{2}} + C$
    5. Restate in terms of x
       $= -2(\cos(x))^{\frac{1}{2}} + C$

12. $\int (1+\sin x)\cos x \; dx$

   1. U-sub appropriate?
      1. Composite function? If there is, I don't see it!
      2. Approx funct/deriv pair? Yes!
         $\underbrace{(1+\sin(x))}_{\text{function}} \rightarrow \underbrace{\cos d(x)}_{\text{deriv}}$
         Let $u = 1+\sin(x)$
   2. Compute du
      $u = 1+\sin(x)$
      $du = \cos(x)\,dx$
   3. Analyze in terms of u and du
      $\int \underbrace{(1+\sin(x))}_{u}\underbrace{\cos(x)\,dx}_{du} = \int u \; du$
   4. Integrate:
      $\int u \; du = \frac{u^2}{2} + C$
   5. Re-express in terms of x
      $\frac{(1+\sin(x))^2}{2} + C$

13. $\frac{d}{dx}\left[\csc\left(\sqrt{x}\right)\right] = \frac{d}{dx}\left[\underset{\text{outer}}{\underset{\uparrow}{\csc}}\left(\underset{\text{inner}}{\underset{\uparrow}{x^{\frac{1}{2}}}}\right)\right]$

$$= \underbrace{-\csc\left(x^{\frac{1}{2}}\right)\cot\left(x^{\frac{1}{2}}\right)}_{\substack{\text{deriv of outer}\\ \text{evaluated at inner}}} \underbrace{\frac{1}{2}x^{-\frac{1}{2}}}_{\text{deriv of inner}} = -\frac{\csc\left(x^{\frac{1}{2}}\right)\cot\left(x^{\frac{1}{2}}\right)}{2x^{\frac{1}{2}}}$$

14. $f(x) = (1+\sec^3 x)^{\frac{1}{2}}$; $f(x) = \underbrace{\left(1+\sec(x)^3\right)}_{[g(x)]^n}{}^{\frac{1}{2}}$

$$f'(x) = \underbrace{\frac{1}{2}\left(1+\sec(x)^3\right)^{-\frac{1}{2}}}_{n[g(x)]^{n-1}} \cdot \frac{d}{dx}\underbrace{\left[1+(\sec(x))^3\right]}_{g'(x)}$$

$$= \frac{1}{2}\left(1+(\sec(x))^3\right)^{-\frac{1}{2}} \underbrace{\left[3(\sec(x))^2\right]}_{n[g(x)]^{n-1}} \underbrace{\frac{d}{dx}\left[\sec(x)\right]}_{g'(x)}$$

$$= \frac{1}{2}\left(1+\sec^3(x)\right)^{-\frac{1}{2}}\left[3\sec^2(x)\right]\left[\sec(x)\tan(x)\right]$$

$$= \frac{3}{2}\frac{\sec^3(x)\tan(x)}{\sqrt{1+\sec^3(x)}}$$

15. $\frac{d}{dx}\left[\underbrace{\cot}_{\text{outer}}\underbrace{\left(x^2+2x\right)}_{\text{inner}}\right] = \underbrace{\left[-\csc^2\left(x^2+2x\right)\right]}_{\substack{\text{deriv of outer}\\ \text{evaluated at inner}}} \underbrace{(2x+2)}_{\text{deriv of inner}}$

16. $\frac{d}{dx}\left[\underbrace{\sin}_{\text{outer}}\underbrace{(\tan(3x))}_{\text{inner}}\right] = \underbrace{[\cos(\tan(3x))]}_{\substack{\text{deriv of outer}\\ \text{evaluated at inner}}} \underbrace{\frac{d}{dx}\left[\underbrace{\tan}_{\text{outer}}\underbrace{(3x)}_{\text{inner}}\right]}_{\text{deriv of inner}}$

$= [\cos(\tan(3x))] \underbrace{\left[\sec^2(3x)\right]}_{\substack{\text{deriv of outer}\\ \text{evaluated at inner}}} \cdot \underbrace{3}_{\text{deriv of inner}} = 3[\cos(\tan(3x))][\sec^2(3x)]$

17. $\int \frac{\tan^2(\sqrt{x})}{\sqrt{x}} dx =$

Remark 1 *Since we don't know how to integrate* $\tan^2(x)$, *we'll convert it to something that we CAN integrate, using the identity* $\tan^2(x) = \sec^2(x) - 1$.

$\int \frac{\tan^2(\sqrt{x})}{\sqrt{x}} dx = \int \left(\sec^2\left(x^{\frac{1}{2}}\right) - 1\right) x^{-\frac{1}{2}} dx$

$= \int \sec^2\left(x^{\frac{1}{2}}\right) \cdot x^{-\frac{1}{2}} dx - \int x^{-\frac{1}{2}} dx$

$= \int \sec^2\left(x^{\frac{1}{2}}\right) \cdot x^{-\frac{1}{2}} dx - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$

1. U-sub appropriate?
   1. Composite Function? Yes!
      $\sec^2\left(x^{\frac{1}{2}}\right)$
      Let $u = x^{\frac{1}{2}}$
   2. Approx funct/deriv pair? Yes!
      $\underbrace{x^{\frac{1}{2}}}_{\text{function}} \rightarrow \underbrace{x^{-\frac{1}{2}}}_{\text{deriv}}$
      Let $u = x^{\frac{1}{2}}$
2. Compute du
   $u = x^{\frac{1}{2}}$
   $\Rightarrow \frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$
   $\Rightarrow du = \frac{1}{2}x^{-\frac{1}{2}}dx$
   $\Rightarrow 2du = x^{-\frac{1}{2}}dx$
3. Analyze in terms of u and du
   $\underbrace{\int \sec^2\left(x^{\frac{1}{2}}\right)}_{\sec^2 u} \underbrace{x^{-\frac{1}{2}}dx}_{2du} - 2x^{\frac{1}{2}} + C = \int \sec^2 u \; 2u \; du - 2x^{\frac{1}{2}} + C$
   $= 2\int \sec^2 u \; du - 2x^{\frac{1}{2}} + C$
4. Integrate
   $= 2\tan(u) - 2x^{\frac{1}{2}} + C$
5. Rewrite in terms of x
   $= 2\tan\left(x^{\frac{1}{2}}\right) - 2x^{\frac{1}{2}} + C$

18. $\int \tan(3x)\sec^2(3x)\,dx$

    (a) The FIRST WAY:

1. Is U-sub appropriate?
   1. Composite function? Yes!
      $\tan(3x)$
      Let $u = \tan(3x)$
   2. Approx funct/deriv pair? Yes!
      $\underbrace{\tan(3x)}_{\text{function}} \rightarrow \underbrace{\sec^2(3x)}_{\text{derivative}}$
      Let $u = \tan(3x)$
2. Compute du
   $u = \tan(3x)$
   $\Rightarrow \frac{du}{dx} = \sec^2(3x) \cdot 3$
   $\Rightarrow du = 3\sec^2(3x)\, dx$
   $\Rightarrow \frac{1}{3}du = \sec^2(3x)\, dx$
3. Analyze in terms of u and du
   $\underbrace{\int \tan(3x)}_{u} \underbrace{\sec^2(3x)\, dx}_{\frac{1}{3}du} = u\frac{1}{3}du = \frac{1}{3}\int u\ du$
4. Integrate
   $\frac{1}{3}\int u\ du = \frac{1}{3}\frac{u^2}{2} + C = \frac{u^2}{6} + C$
5. Rewrite in terms of x
   $= \frac{\tan^2(3x)}{6} + C$

(b) The SECOND WAY
$\int \tan(3x)\sec^2(3x)\, dx = \int \sec(3x)\sec(3x)\tan(3x)$

1. Is U-sub appropriate?
   1. Composite function??? I there is, I don't see it!
      Let $u =???$
   2. Approx. function/deriv pair? Yes!
      $\sec(3x) \leftrightarrow \sec(3x)\tan(3x)$
      Let $u = \sec(3x)$
2. Compute du
   $u = \sec(3x)$
   $\Rightarrow \frac{du}{dx} \sec(3x)\tan(3x) \cdot 3$
   $\Rightarrow du = 3\sec(3x)\tan(3x)$
   $\Rightarrow \frac{1}{3}du = \sec(3x)\tan(3x)$
3. Analyze in terms of u and du
   $\underbrace{\int \sec(3x)}_{u} \underbrace{\sec(3x)\tan(3x)\, dx}_{\frac{1}{3}du} = \frac{1}{3}\int u\ du$
4. Integrate $\frac{1}{3}\int u\ du = \frac{1}{3}\ \frac{u^2}{2} + C = \frac{u^2}{6} + C$
5. Re-express in terms of x.
   $= \frac{\sec^2(3x)}{6} + C$

19. $\int \tan^7(x)\sec^2(x)\, dx = \int (\tan x)^7 \sec^2(x)\, dx$
   1. U-sub appropriate?

1. Composite function? Yes!
   $(\tan x)^7$
   Let $u = \tan x$
2. Approx funct/deriv pair? Yes!
   $\underbrace{\tan(x)}_{\text{funct}} \rightarrow \underbrace{\sec^2(x)}_{\text{deriv}}$
   Let $u = \tan(x)$

2. Compute du
   $u = \tan(x)$
   $\Rightarrow \frac{du}{dx} = \sec^2(x)$
   $\Rightarrow du = \sec^2(x)\, dx$
3. Analyze in terms of u and du
   $\underbrace{\int (\tan(x))^7}_{u^7} \underbrace{\sec^2(x)\, dx}_{du} = \int u^7\ du$
4. Integrate
   $\int u^7\ du = \frac{u^8}{8} + C$
5. Re-express in terms of x
   $\frac{(\tan(x))^8}{8} + C$

20. $\underbrace{\int \sec^3(x)\tan(x)\, dx = \int (\sec(x))^2 \sec(x)\tan(x)\, dx}_{\text{Get this trick!}}$

1. U-sub appropriate?
   1. Composite function? Yes!
      $(\sec x)^2$
      Let $u = \sec(x)$
   2. Approx funct/deriv pair? Yes!
      $\underbrace{\sec(x)}_{\text{funct.}} \rightarrow \underbrace{\sec(x)\tan(x)}_{\text{deriv.}}$
      Let $u = \sec(x)$
2. Compute du
   $u = \sec(x)$
   $\Rightarrow \frac{du}{dx} = \sec(x)\tan(x)$
   $\Rightarrow du = \sec(x)\tan(x)\, dx$
3. Analyze in terms of u and du
   $\int \underbrace{(\sec(x))^2}_{u^2} \underbrace{\sec(x)\tan(x)}_{du}\, dx = \int u^2 du$
4. Integrate
   $\int u^2 du = \frac{u^3}{3} + C$
5. Re-express in terms of u and du
   $= \frac{(\sec(x))^3}{3} + C = \frac{1}{3}\sec^3(x) + C$