MTH 1125 - Test 2 (12pm Class) - Solutions

Fall 2023

Pat Rossi Name ______________________________

Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\frac{d}{dx}\left[8x^5+10x^4+8x^3+12x^2+12x+24\sqrt{x}+10\right]=$

$$\frac{d}{dx}\left[8x^5+10x^4+8x^3+12x^2+12x+24x^{\frac{1}{2}}+10\right]$$

$$=8\left[5x^4\right]+10\left[4x^3\right]+8\left[3x^2\right]+12\left[2x\right]+12+24\left[\tfrac{1}{2}x^{-\frac{1}{2}}\right]+0$$

$$=40x^4+40x^3+24x^2+24x+12+12x^{-\frac{1}{2}}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[8x^5+10x^4+8x^3+12x^2+12x+24x^{\frac{1}{2}}+10\right]=40x^4+40x^3+24x^2+24x+12+12x^{-\frac{1}{2}}}$$

2. Compute: $\frac{d}{dx}\left[\left(\sin(x)+\cos(x)\right)\left(8x^5+4x+2\right)\right]=$

$$\frac{d}{dx}\left[\underbrace{\left(\sin(x)+\cos(x)\right)}_{1^{st}}\underbrace{\left(8x^5+4x+2\right)}_{2^{nd}}\right]=\underbrace{\left(\cos(x)-\sin(x)\right)}_{1^{\text{st}}\text{ prime}}\cdot\underbrace{\left(8x^5+4x+2\right)}_{2^{nd}}+\underbrace{\left(40x^4+4\right)}_{2^{\text{nd}}\text{ prime}}\cdot\underbrace{\left(\sin(x)+\cos(x)\right)}_{1^{st}}$$

$$\boxed{\frac{d}{dx}\left[\left(\sin(x)+\cos(x)\right)\left(8x^5+4x+2\right)\right]=\left(\cos(x)-\sin(x)\right)\left(8x^5+4x+2\right)+\left(40x^4+4\right)\left(\sin(x)+\cos(x)\right)}$$

3. Compute: $\frac{d}{dx}\left[\frac{3x^5+5x^3+15x}{4x^2+8}\right]=$

$$\frac{d}{dx}\left[\frac{\overbrace{3x^5+5x^3+15x}^{\text{top}}}{\underbrace{4x^2+8}_{\text{Bottom}}}\right]=\frac{\overbrace{\left(15x^4+15x^2+15\right)}^{\text{top prime}}\cdot\overbrace{\left(4x^2+8\right)}^{\text{bottom}}-\overbrace{8x}^{\text{bottom prime}}\cdot\overbrace{\left(3x^5+5x^3+15x\right)}^{\text{top}}}{\underbrace{\left(4x^2+8\right)^2}_{\text{bottom squared}}}$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[\frac{3x^5+5x^3+15x}{4x^2+8}\right]=\frac{\left(15x^4+15x^2+15\right)\left(4x^2+8\right)-8x\left(3x^5+5x^3+15x\right)}{\left(4x^2+8\right)^2}\ :\ 36x^6+140x^4+60x^2+120}$$

4. Compute: $\frac{d}{dx}\left[\left(3x^3+6x^2+15x\right)^{10}\right] =$ This is the derivative of a function raised to a power.

$$\frac{d}{dx}\left[\left(3x^3+6x^2+15x\right)^{10}\right] = \underbrace{10\left(3x^3+6x^2+15x\right)^9}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(9x^2+12x+15\right)}_{\substack{\text{derivative}\\ \text{of inner}}}$$

i.e., $\frac{d}{dx}\left[\left(3x^3+6x^2+15x\right)^{10}\right] = 10\left(3x^3+6x^2+15x\right)^9\left(9x^2+12x+15\right)$

5. Given that $f(x) = 2x^2 - 3x + 2$, give the *equation* of the line tangent to the graph of $f(x)$ at the point $(2,4)$.

We need two things:

   i. A **point** on the line (We have that: $(x_1, y_1) = (2,4)$)

   ii. The **slope** of the line (This is $f'(x_1)$)

   $f'(x) = 4x - 3$

At the point $(x_1, y_1) = (2,4)$, **the slope is** $f'(2) = 4(2) - 3 = 5$

We will use the Point-Slope equation of a line:

$(y - y_1) = m(x - x_1)$ (Where m is the slope and (x_1, y_1) is a known point on the line.)

Thus, the equation of the line tangent to the graph of $f(x)$ is:

$(y - 4) = 5(x - 2)$

The equation of the line tangent is $(y - 4) = 5(x - 2)$

6. Given that $w = \cos(u)$ and that $u = 4t^2 + 2t + 2$; compute $\frac{dw}{dt}$ **using the Liebniz form of the Chain Rule.** (In particular, when doing this exercise, *write explicitly the Liebniz form of the chain rule that you are going to use.)*

We know:

$\frac{dw}{du} = -\sin(u)$

$\frac{du}{dt} = 8t + 2$

We want: $\frac{dw}{dv}$

By the Liebniz form of the Chain Rule:

$$\frac{dw}{dt} = \frac{dw}{du}\frac{du}{dt} = -\sin(u)(8t+2) = \underbrace{-\sin\left(4t^2+2t+2\right)(8t+2)}_{\substack{\text{express solely in terms of} \\ \text{independent variable } v}}$$

i.e. $\frac{dw}{dt} = -\sin(u)(8t+2) = -\sin(4t^2+2t+2)(8t+2)$

7. Compute: $\frac{d}{dx}\left[\sec(8x^5+5x^2)\right] =$

Outer:	=	sec ( )
Deriv. of outer	=	sec ( ) tan ( )

$$\frac{d}{dx}\left[\begin{array}{c} \sec\left(\underbrace{8x^5+5x^2}\right) \\ \uparrow \qquad \uparrow \\ \text{outer} \quad \text{inner} \end{array}\right] = \underbrace{\sec\left(8x^5+5x^2\right)\tan\left(8x^5+5x^2\right)}_{\substack{\text{Deriv of outer,} \\ \text{eval. at inner}}} \cdot \underbrace{\left(40x^4+10x\right)}_{\substack{\text{deriv. of} \\ \text{inner}}}$$

i.e., $\frac{d}{dx}\left[\sec(8x^5+5x^2)\right] = \sec(8x^5+5x^2)\tan(8x^5+5x^2)(40x^4+10x)$

8. Compute: $\frac{d}{dx}\left[\left(\frac{3x^2+12x}{4x^2+8x+16}\right)^8\right] =$ In the broadest sense, this is the derivative of a *function raised to a power* - USE the CHAIN RULE.

$$\frac{d}{dx}\left[\underbrace{\left(\frac{3x^2+12x}{4x^2+8x+16}\right)^8}_{(g(x))^n}\right] = \underbrace{8\left(\frac{3x^2+12x}{4x^2+8x+16}\right)^7}_{\substack{\text{power rule}\\ \text{as usual}}} \cdot \underbrace{\left(\frac{d}{dx}\left[\frac{3x^2+12x}{4x^2+8x+16}\right]\right)}_{\substack{\text{deriv of}\\ \text{inner Function}}}$$

$$= 8\left(\frac{3x^2+12x}{4x^2+8x+16}\right)^7 \underbrace{\frac{(6x+12)(4x^2+8x+16) - (8x+8)(3x^2+12x)}{(4x^2+8x+16)^2}}_{\substack{\text{quotient}\\ \text{rule}}}$$

i.e., $\frac{d}{dx}\left[\left(\frac{3x^2+12x}{4x^2+8x+16}\right)^8\right] = 8\left(\frac{3x^2+12x}{4x^2+8x+16}\right)^7 \frac{(6x+12)\left(4x^2+8x+16\right) - (8x+8)\left(3x^2+12x\right)}{(4x^2+8x+16)^2}$

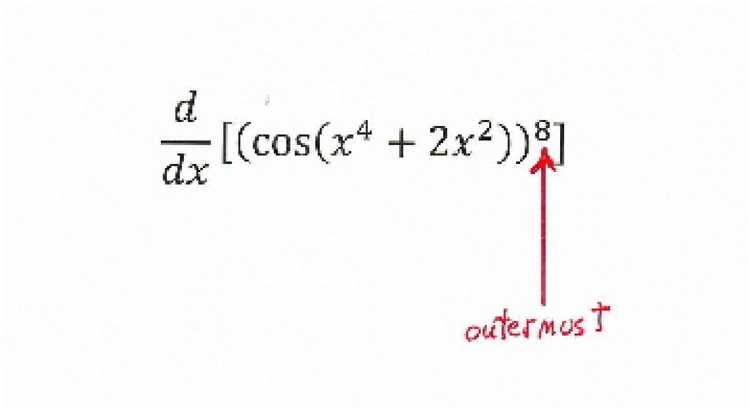
9. Compute: $\frac{d}{dx}\left[\cos^8\left(x^4+2x^2\right)\right] =$

Let's rewrite this:

$\frac{d}{dx}\left[\left(\cos\left(x^4+2x^2\right)\right)^8\right]$

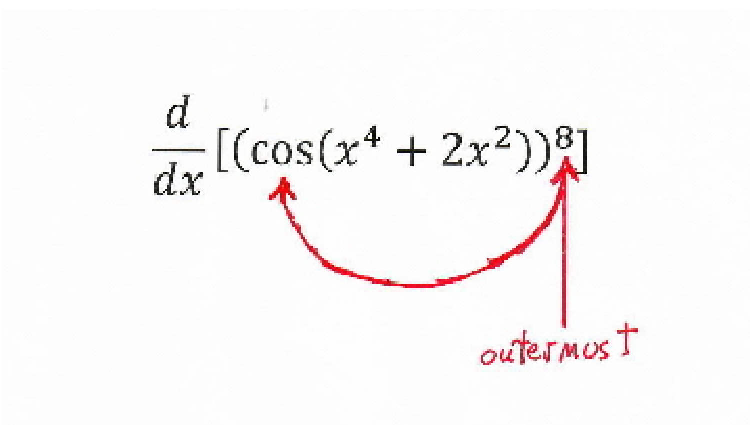
This is the composition of *three* functions.

Differentiate the outermost function and evaluate it at everything inside



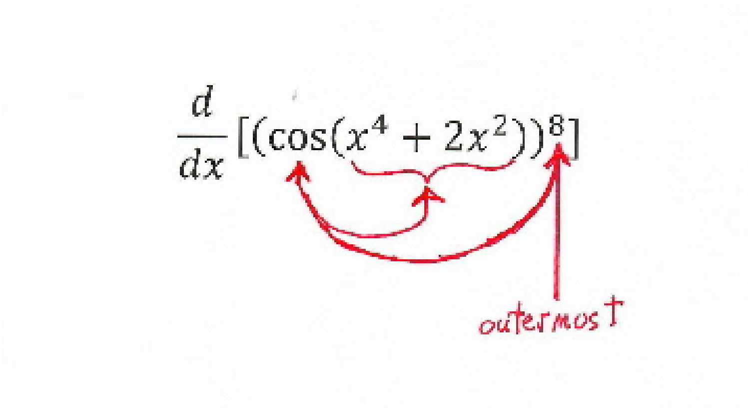
This yields: $8\left(\cos\left(x^4+2x^2\right)\right)^7$

Next: Multiply by the derivative of the next outermost function and evaluate it at everything inside of it.



This yields: $8\left(\cos\left(x^4+2x^2\right)\right)^7 \cdot \left(-\sin\left(x^4+2x^2\right)\right)$

Finally: Multiply by the derivative of the innermost function.



This yields: $8\left(\cos\left(x^4+2x^2\right)\right)^7 \cdot \left(-\sin\left(x^4+2x^2\right)\right) \cdot \left(4x^3+4x\right)$

$$\begin{aligned} \text{i.e., } \frac{d}{dx}\left[\cos^8\left(x^4+2x^2\right)\right] &= 8\left(\cos\left(x^4+2x^2\right)\right)^7 \cdot \left(-\sin\left(x^4+2x^2\right)\right) \cdot \left(4x^3+4x\right) \\ &= -8\left(\cos\left(x^4+2x^2\right)\right)^7 \cdot \sin\left(x^4+2x^2\right) \cdot \left(4x^3+4x\right) \end{aligned}$$

Alternatively:

Re-Write!

$\frac{d}{dx}\left[\cos^8\left(x^4+2x^2\right)\right] = \frac{d}{dx}\left[\left(\cos\left(x^4+2x^2\right)\right)^8\right]$

In the broadest sense, this is *the derivative of a function raised to a power*

$$\begin{aligned} \frac{d}{dx}\left[\left(\cos\left(x^4+2x^2\right)\right)^8\right] &= \underbrace{8\left(\cos\left(x^4+2x^2\right)\right)^7}_{\text{power rule as usual}} \cdot \underbrace{\left(\frac{d}{dx}\left[\cos\left(x^4+2x^2\right)\right]\right)}_{\text{derivative of inner}} \\ &= 8\left(\cos\left(x^4+2x^2\right)\right)^7 \cdot \underbrace{\left[-\sin\left(x^4+2x^2\right) \cdot \left(4x^3+4x\right)\right]}_{\text{Chain Rule}} \end{aligned}$$

$$\text{i.e., } \frac{d}{dx}\left[\cos^8\left(x^4+2x^2\right)\right] = -8\left(\cos\left(x^4+2x^2\right)\right)^7 \cdot \sin\left(x^4+2x^2\right) \cdot \left(4x^3+4x\right)$$

10. Given that $5x^4 - x^4y^4 = \cos(y)$, compute $\frac{dy}{dx}$

i. Differentiate both sides w.r.t. x

$$\frac{d}{dx}\left[5x^4 - \underbrace{x^4}_{1^{\text{st}}}\underbrace{y^4}_{2^{\text{nd}}}\right] = \frac{d}{dx}\left[\cos(y)\right]$$

$$\Rightarrow 20x^3 - \left(\underbrace{4x^3}_{1^{\text{st}}\text{ prime}} \cdot \underbrace{y^4}_{2^{\text{nd}}} + \underbrace{4y^3\frac{dy}{dx}}_{2^{\text{nd}}\text{ prime}} \cdot \underbrace{x^4}_{1^{\text{st}}}\right) = -\sin(y) \cdot \frac{dy}{dx}$$

Simplifying, we have:

$$20x^3 - 4x^3y^4 - 4x^4y^3\frac{dy}{dx} = -\sin(y) \cdot \frac{dy}{dx}$$

ii. Solve algebraically for $\frac{dy}{dx}$

a. Get $\frac{dy}{dx}$ terms on left side, all other terms on right side

$$\Rightarrow -4x^4y^3\frac{dy}{dx} + \sin(y) \cdot \frac{dy}{dx} = -20x^3 + 4x^3y^4$$

b. Factor out $\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx}\left(-4x^4y^3 + \sin(y)\right) = -20x^3 + 4x^3y^4$$

c. Divide both sides by the cofactor of $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-20x^3+4x^3y^4}{-4x^4y^3+\sin(y)} = \frac{20x^3-4x^3y^4}{4x^4y^3-\sin(y)}$$

$$\boxed{\frac{dy}{dx} = \frac{-20x^3+4x^3y^4}{-4x^4y^3+\sin(y)} = \frac{20x^3-4x^3y^4}{4x^4y^3-\sin(y)}}$$

11. Given that $f(x) = 5x^2 - 6x + 2$, compute $f'(x)$ **using the definition of derivative.** (i.e., using the "limit process.")

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[5(x+\Delta x)^2 - 6(x+\Delta x) + 2\right] - \left[5x^2 - 6x + 2\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[5\left(x^2 + 2x\Delta x + \Delta x^2\right) - 6(x+\Delta x) + 2\right] - \left[5x^2 - 6x + 2\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[5x^2 + 10x\Delta x + 5\Delta x^2 - 6x - 6\Delta x + 2\right] - \left[5x^2 - 6x + 2\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{10x\Delta x + 5\Delta x^2 - 6\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(10x + 5\Delta x - 6)}{\Delta x} = \lim_{\Delta x \to 0} (10x + 5\Delta x - 6) = 10x + 5(0) - 6 = 10x - 6$$

$$\boxed{\text{i.e., } f'(x) = 10x - 6}$$

Extra (Wow! 10 Points)

Given that $S'(x) = \frac{1}{\sqrt{1-x^2}}$ (i.e., $\frac{d}{dx}[S(x)] = \frac{1}{\sqrt{1-x^2}}$); compute $\frac{d}{dx}[S(\cos(x))]$

Outer:	=	$S(\ \ )$
Deriv. of outer	=	$\frac{1}{\sqrt{1-(\ \ )^2}}$

$$\frac{d}{dx}\left[S\left(\underbrace{\cos(x)}\right)\right] = \underbrace{\frac{1}{\sqrt{1-(\cos(x))^2}}}_{\text{Deriv of outer, eval. at inner}} \cdot \underbrace{(-\sin(x))}_{\text{deriv. of inner}} = -\frac{\sin(x)}{\sqrt{1-\cos^2(x)}} = -\frac{\sin(x)}{\sqrt{\sin^2(x)}} = -\frac{\sin(x)}{|\sin(x)|}$$

↑ outer ↑ inner

$$\boxed{\text{i.e., } \frac{d}{dx}[S(\cos(x))] = -\frac{\sin(x)}{\sqrt{1-\cos^2(x)}} = -\frac{\sin(x)}{|\sin(x)|}}$$