MTH 1125 (2 pm) Test #3 - Solutions

Fall 2021

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Instructions. Show CLEARLY how you arrive at your answers.

1. $f(x) = x^3 - 6x^2 - 15x + 1$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

   1. Compute $f'(x)$ and find the critical numbers

      $f'(x) = 3x^2 - 12x - 15$

      a. "Type a" $(f'(c) = 0)$

         Set $f'(x) = 0$ and solve for x

         $\Rightarrow f'(x) = 3x^2 - 12x - 15 = 0$

         $\Rightarrow 3x^2 - 12x - 15 = 0$

         $\Rightarrow x^2 - 4x - 5 = 0$

         $\Rightarrow (x+1)(x-5) = 0$

         $\Rightarrow x = -1$ and $x = 5$ are critical numbers.

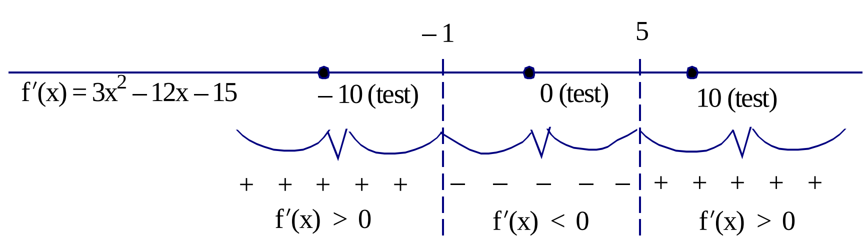
      b. "Type b" ($f'(c)$ is undefined)

         Look for x-value that causes division by zero.

         No "type b" critical numbers

   2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

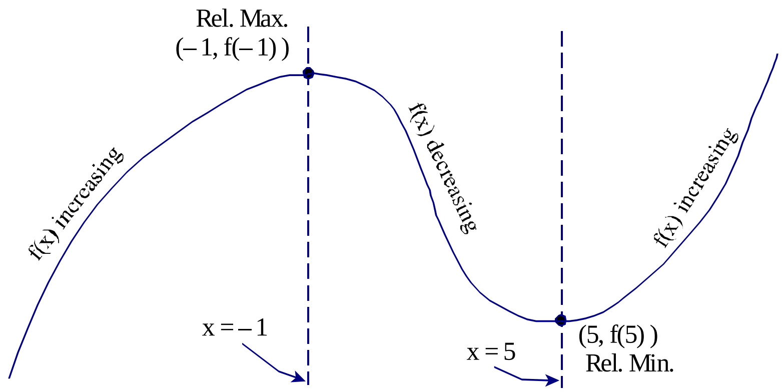
   3. Pick a "test point" from each interval to plug into $f'(x)$



$f(x)$ is **increasing** on the interval(s) $(-\infty, -1)$ and $(5, \infty)$
(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-1, 5)$
(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(-1, f(-1)) = (-1, 9)$

Rel Min $(5, f(5)) = (5, -99)$

2. $f(x) = \frac{1}{4}x^4 - 2x^3 + 3$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection. (Caution - there are **two** points if inflection. Make sure you get them both!)

   i. Compute $f''(x)$ and find possible points of inflection

   $f'(x) = x^3 - 6x^2$

   $f''(x) = 3x^2 - 12x$

   a. "Type a" $(f''(c) = 0)$

   Set $f''(x) = 3x^2 - 12x = 0$

   $\Rightarrow 3x^2 - 12x = 0$

   $\Rightarrow x^2 - 4x = 0$

   $\Rightarrow x(x-4) = 0$

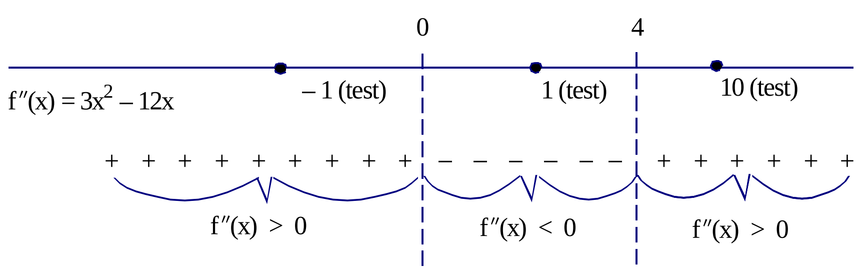
   $\Rightarrow x = 0$; and $x = 4$ possible points of inflection

   b. "Type b" $(f''(c)$ undefined)

   There are none.

   ii. Draw a sign graph of $f''(x)$, using the possible points of inflection to partition the x-axis

   iii. From each interval select a "test point" to plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-\infty, 0)$ and $(4, \infty)$

(Because $f''(x) > 0$ on these intervals)

$f(x)$ is **concave down** on the interval $(0, 4)$

(Because $f''(x) < 0$ on this interval)

Since $f(x)$ changes concavity at $x = 0$ and $x = 40$, the points:

$(0, f(0)) = (0, 3)$

and $(4, f(4)) = (4, -61)$ *are* points of inflection.

3. $f(x) = 2x^3 + 5x^2 - 4x + 2$ on the interval $[-3, 0]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

Since $f(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0, 3]$, we can use the Absolute Max/Min Value Test

i. Compute $f'(x)$ and find the critical numbers

$f'(x) = 6x^2 + 10x - 4$

a. "Type a" ($f'(c) = 0$)

Set $f'(x) = 6x^2 + 10x - 4 = 0$

$\Rightarrow 6x^2 + 10x - 4 = 0$

$\Rightarrow 3x^2 + 5x - 2 = 0$

$\Rightarrow (x+2)(3x-1) = 0$

$x + 2 = 0$ or $3x + 1 = 0$

$\Rightarrow x = -2;\ x = \frac{1}{3}$

Since $\frac{1}{3} \notin [-3, 0]$, we discard it as a critical number

$\Rightarrow x = -2$ "type a" crit. number

Alternatively: If we couldn't figure out how to factor the equation

$6x^2 + 10x - 4 = 0$, we could use the Quadratic Equation to find the solutions.

The solutions to $ax^2 + bx + c = 0$ are given by the formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Thus, the solution to the equation $6x^2 + 10x - 4 = 0$ are given by

$x = \frac{-10 \pm \sqrt{(10)^2 - 4(6)(-4)}}{2(6)} = \frac{-10 \pm \sqrt{196}}{12} = -2, \frac{1}{3}$

i.e., $x = -2;\ x = \frac{1}{3}$

Since $\frac{1}{3} \notin [-3, 0]$, we discard it as a critical number

$\Rightarrow x = -2$ "type a" crit. number

b. "Type b" ($f'(c)$ undefined)

There are none.

ii. Plug critical numbers and endpoints into the original function.

$f(-3) = 2(-3)^3 + 5(-3)^2 - 4(-3) + 2 = 5$

$f(-2) = 2(-2)^3 + 5(-2)^2 - 4(-2) + 2 = 14 \leftarrow$ Abs Max Value

$f(0) = 2(0)^3 + 5(0)^2 - 4(0) + 2 = 2 \leftarrow$ Abs Max Value

Abs Max Value = 14
(attained at $x = -2$)

Abs Min Value = 2
(attained at $x = 0$)

4. $f(x) = \frac{1}{3}x^{\frac{12}{5}} - 8x^{\frac{2}{5}} + 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

   1. Compute $f'(x)$ and find the critical numbers

      $f'(x) = \frac{4}{5}x^{\frac{7}{5}} - \frac{16}{5}x^{-\frac{3}{5}} = \frac{4x^{\frac{7}{5}}}{5} - \frac{16}{5x^{\frac{3}{5}}} = \frac{4x^{\frac{7}{5}}}{5}\frac{x^{\frac{3}{5}}}{x^{\frac{3}{5}}} - \frac{16}{5x^{\frac{3}{5}}} = \frac{4x^2-16}{5x^{\frac{3}{5}}}$

      i.e., $f'(x) = \frac{4x^2-16}{5x^{\frac{3}{5}}}$

      a. "Type a" $(f'(c) = 0)$

         Set $f'(x) = 0$ and solve for x

         $\Rightarrow f'(x) = \frac{4x^2-16}{5x^{\frac{3}{5}}} = 0$

         $\Rightarrow 4x^2 - 16 = 0$

         $\Rightarrow x^2 - 4 = 0$

         $\Rightarrow (x+2)(x-2) = 0$

         $\Rightarrow x = -2$ and $x = 2$ are critical numbers.

      b. "Type b" $(f'(c)$ is undefined$)$

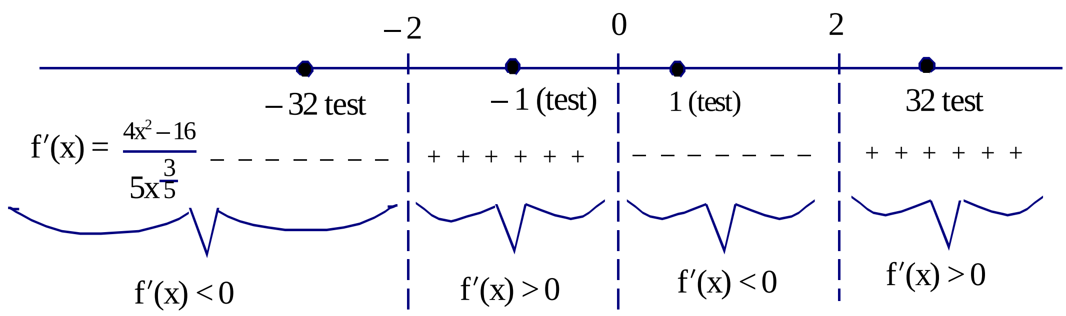
         Look for x-value that causes division by zero.

         $\Rightarrow 5x^{\frac{3}{5}} = 0$

         $\Rightarrow x = 0$ "type b" critical number

   2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

   3. Pick a "test point" from each interval to plug into $f'(x)$



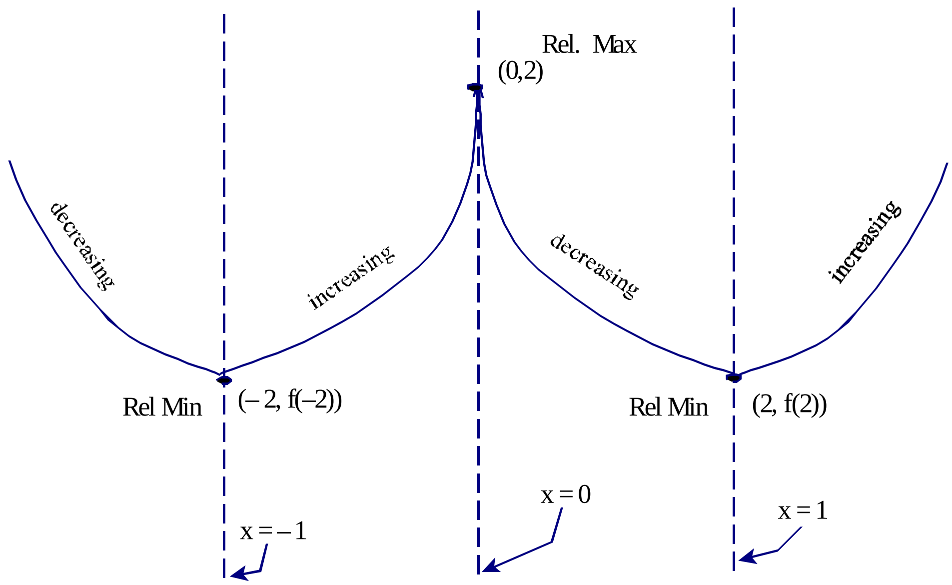
$f(x)$ is **increasing** on the interval(s) $(-2, 0)$ and $(2, \infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-\infty, -2)$ and $(0, 2)$

(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.

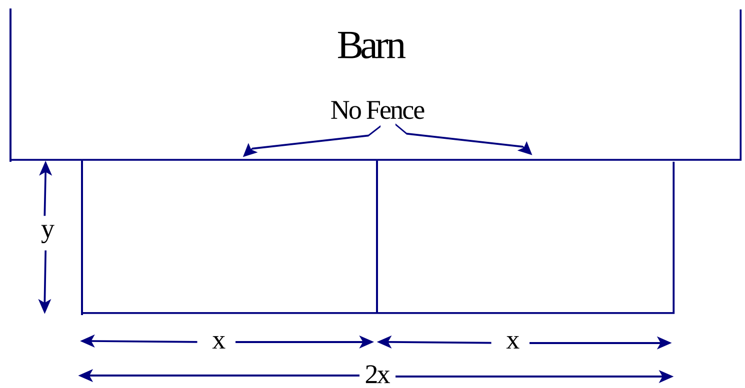


Rel Minimums: $(-2, f(-2)) = \left(-2, \frac{1}{3}(-2)^{\frac{12}{5}} - 8(-2)^{\frac{2}{5}} + 2\right)$

and $(2, f(2)) = \left(2, \frac{1}{3}(2)^{\frac{12}{5}} - 8(2)^{\frac{2}{5}} + 2\right)$

Rel Maximum: $(0, f(0)) = (0, 2)$

5. Farmer Joe has 180 feet of wire fencing. He will use the fencing to make a rectangular pen. His barn will form one side of the pen, so no wire fencing will be used on that side. In addition, he will use some of the fencing to partition the pen into two smaller pens of similar shape and equal area. (See below) What should the overall dimensions of the pen be, in order for the enclosed area to be as large as possible?



1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = 2xy$

   a. Draw a picture where relevant.

      (Done)

2. Express A as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that Joe will use exactly 180 feet of wire fencing

   Note that Joe will use 1 piece of fence of length $2x$ and three pieces of fence of length y.
   Hence, $2x + 3y = 180\text{ft}$

   $\Rightarrow 2x = 180\text{ft } - 3y$

   Plug this into the equation $A = 2xy$, we have:

   $A = (180\text{ft} - 3y)\, y = 180\text{ft } y - 3y^2$

   $\Rightarrow A\,(y) = 180\text{ft } y - 3y^2$

3. Determine the restrictions on the independent variable y.

   From the picture, $0\text{ft} \leq y \leq 60\text{ft}$

4. Maximize $A(y)$, using the techniques of Calculus.

   Note that $A(y)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0\text{ft}, 60\text{ft}]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(y) = 180\text{ft} - 6y$

      a. "Type a" ($A'(c) = 0$)

         $\Rightarrow A'(y) = 180\text{ft} - 6y = 0$

         $\Rightarrow 180\text{ft} - 6y = 0$

         $\Rightarrow -6y = -180\text{ft}$

         $\Rightarrow y = 30\text{ft}$ critical number

      b. "Type b" ($A'(c)$ is undefined)

         Look for y-values that cause division by zero in $A'(y)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0\text{ft}) = 180\text{ft}(0\text{ft}) - 3(0\text{ft})^2 = 0\text{ft}^2$

      $A(30\text{ft}) = 180\text{ft}(30\text{ft}) - 3(30\text{ft})^2 = 2700\text{ft}^2 \longleftarrow$ Abs Max Value

      $A(60\text{ft}) = 180\text{ft}(60\text{ft}) - 3(60\text{ft})^2 = 0\text{ft}^2$

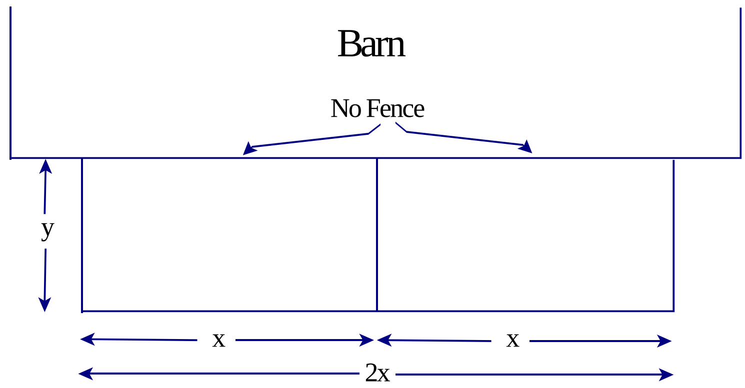
5. Make sure that we've answered the original question.

   1. "What should the overall dimensions of the pen be ..."

      Length $2x = 180\text{ft} - 3y = 180\text{ft} - 3(30\text{ft}) = 90\text{ft}$

Length $2x = 90\text{ft}$
Width $y = 30\text{ft}$

Alternative Solution appears on the next page



6.

1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = 2xy$

   a. Draw a picture where relevant.

      (Done)

2. Express A as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that Joe will use exactly 150 feet of wire fencing

   Note that Joe will use 1 piece of fence of length $2x$ and three pieces of fence of length y.
   Hence, $2x + 3y = 180\text{ft}$

   $\Rightarrow 3y = 180\text{ft} - 2x$

   $\Rightarrow y = 60\text{ft} - \frac{2}{3}x$

   Plug this into the equation $A = 2xy$, we have:

   $A = 2x\left(60\text{ft} - \frac{2}{3}x\right) = 120\text{ft}\ x - \frac{4}{3}x^2$

   $\Rightarrow A(x) = 120\text{ft}\ x - \frac{4}{3}x^2$

3. Determine the restrictions on the independent variable x.

   From the picture, $0\text{ft} \leq x \leq 90\text{ft}$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0\text{ft}, 90\text{ft}]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = 120\text{ft} - \frac{8}{3}x$

      a. "Type a" $(A'(c) = 0)$

         $\Rightarrow A'(x) = 120\text{ft} - \frac{8}{3}x = 0$

         $\Rightarrow 120\text{ft} - \frac{8}{3}x = 0$

         $\Rightarrow -\frac{8}{3}x = -120\text{ft}$

         $\Rightarrow x = \frac{360}{8}\text{ft}$

         $\Rightarrow x = 45\text{ft}$ critical number

      b. "Type b" $(A'(c)$ is undefined)

         Look for x-values that cause division by zero in $A'(x)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0\text{ft}) = 120\text{ft}\,(0\text{ft}) - \frac{4}{3}(0\text{ft})^2 = 0\text{ft}^2$

      $A(45\text{ft}) = 120\text{ft}\,(45\text{ft}) - \frac{4}{3}(45\text{ft})^2 = 2700\text{ft}^2 \longleftarrow$ Abs Max Value

      $A(90\text{ft}) = 120\text{ft}\,(90\text{ft}) - \frac{4}{3}(90\text{ft})^2 = 0\text{ft}^2$

5. Make sure that we've answered the original question.

   1. "What should the overall dimensions of the pen be ..."

      Length $2x = 2(45\text{ft}) = 90\text{ft}$

      Width $y = 60\text{ft} - \frac{2}{3}x = 60\text{ft} - \frac{2}{3}(45\text{ft}) = 30\text{ft}$

      Length $2x = 90\text{ft}$

      Width $y = 30\text{ft}$