MTH 3311 Test #1 - Solutions

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Show CLEARLY how you arrive at your answers.

1. Classify the following according to **order** and **linearity.** If an equation is **not linear,** explain why.

   (a) $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + \sin(x)\,y = \tan(x)$

   Order 2; Linear

   (b) $y^{(6)} + 6y''' - 2yy' + 6y = e^x$

   Order 6; Non-Linear. The funtions y and y' appear as cofactors in the same term.

   (c) $8x^4y'' + \cos(x)\,y' + 3y = \sec(6x)$

   Order 2; Linear

   (d) $3x^4\frac{d^2y}{dx^2} + 3y^4\frac{dy}{dx} + 6y = \frac{1}{x^2+1}$

   Order 2; Linear. y is raised to a power other than 1. Also, the funtions y^4 and $\frac{dy}{dx}$ appear as cofactors in the same term. (either reason is sufficient.)

2. Solve: $\frac{dy}{dx} = \frac{x\sqrt{x^2+5}}{y^2}$; subject to the initial condition $y(2) = 3$

Separate!

$\Rightarrow y^2 \frac{dy}{dx} = x\sqrt{x^2+5}$

$\Rightarrow y^2 dy = (x^2+5)^{\frac{1}{2}}\, x\, dx$

$$\Rightarrow \int y^2 dy = \int \underbrace{\left(x^2+5\right)^{\frac{1}{2}}}_{u^{\frac{1}{2}}} \underbrace{x\,dx}_{\frac{1}{2}du} = \frac{1}{2}\int u^{\frac{1}{2}} du$$

$\Rightarrow \frac{y^3}{3} = \frac{1}{2}\frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + C = \frac{1}{3}u^{\frac{3}{2}} + C$

$\Rightarrow y^3 = u^{\frac{3}{2}} + C_1$

$\Rightarrow y^3 = (x^2+5)^{\frac{3}{2}} + C_1$

Recall: $y(2) = 3$

$\Rightarrow (3)^3 = \left((2)^2+5\right)^{\frac{3}{2}} + C_1 \Rightarrow (3)^3 = 9^{\frac{3}{2}} + C_1$

$\Rightarrow 27 = 27 + C$

$\Rightarrow 0 = C$

Hence, $y^3 = (x^2+5)^{\frac{3}{2}}$

$\Rightarrow y = \sqrt{x^2+5}$

3. Solve: $y' + x^{-2}y = 6x^{-2}$, using the "Integrating Factor" Method. (Assume $y > 0$)

1. Re-write the equation in the form: $y' + P(x)\,y = Q(x)$

$$y' + \underbrace{x^{-2}}_{P(x)} y = \underbrace{6x^{-2}}_{Q(x)}$$

2. Compute the integrating factor:

$$e^{\int P(x)dx} = e^{\int x^{-2}dx} = e^{-x^{-1}}$$

i.e., $e^{\int P(x)dx} = e^{-x^{-1}}$

3. Multiply both sides by the integrating factor

$$e^{-x^{-1}}y' + x^{-2}e^{-x^{-1}}y = 6x^{-2}e^{-x^{-1}}$$

4. Express the left hand side as the derivative of a product

$$\underbrace{e^{-x^{-1}}}_{1^{\text{st}}} \underbrace{y'}_{2^{\text{nd}}\text{ prime}} + \underbrace{y}_{2^{\text{nd}}} \underbrace{x^{-2}e^{-x^{-1}}}_{1^{\text{st}}\text{ prime}} = 6x^{-2}e^{-x^{-1}}$$

$$\Rightarrow \frac{d}{dx}\left[e^{-x^{-1}}y\right] = 6x^{-2}e^{-x^{-1}}$$

5. Integrate both sides w.r.t. x.

$$\int \frac{d}{dx}\left[e^{-x^{-1}}y\right]dx = \int 6x^{-2}e^{-x^{-1}}dx$$

$$\Rightarrow e^{-x^{-1}}y = 6\int \underbrace{e^{-x^{-1}}}_{e^u}\underbrace{x^{-2}dx}_{du} = 6e^u + C$$

i.e., $e^{-x^{-1}}y = 6e^{-x^{-1}} + C$

6. Solve for y

$$\Rightarrow y = 6 + Ce^{x^{-1}}$$

Our solution is $y = 6 + Ce^{x^{-1}}$

4. Show that the function $y = e^{2x} + 2x^2 + 6x$ is a solution of the differential equation:

$y'' - y' - 2y = -4x^2 - 16x - 2$

Observe:

$y = e^{2x} + 2x^2 + 6x$

$y' = 2e^{2x} + 4x + 6$

$y'' = 4e^{2x} + 4$

Plugging these into the expression $y'' - y' - 2y$, we have:
$y'' - y' - 2y = (4e^{2x} + 4) - (2e^{2x} + 4x + 6) - 2\left(e^{2x} + 2x^2 + 6x\right)$

i.e., $y'' - y' - 2y = -4x^2 - 16x - 2$

Hence, $y = e^{2x} + 2x^2 + 6x$ is a solution of the equation:

$y'' - y' - 2y = -4x^2 - 16x - 2$

5. Determine whether or not the equation is exact. If the equation is exact, solve it.

$(3x^2 + \ln(y) + 3x^2y^3)\, dx + \left(\frac{x}{y} + 3x^3y^2 + \cos(y)\right) dy = 0$

This equation can be analyzed as:

$$\underbrace{(3x^2 + \ln(y) + 3x^2y^3)}_{M(x,y)} dx + \underbrace{\left(\frac{x}{y} + 3x^3y^2 + \cos(y)\right)}_{N(x,y)} dy = 0$$

By convention, we let $M(x, y)$ be the co-factor of dx and we let $N(x, y)$ be the co-factor of dy.

i.e., $M(x, y) = 3x^2 + \ln(y) + 3x^2y^3$ and $N(x, y) = \frac{x}{y} + 3x^3y^2 + \cos(y)$

The Differential equation is **exact** precisely when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Observe: $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[3x^2 + \ln(y) + 3x^2y^3\right] = \frac{1}{y} + 9x^2y^2$

Also: $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{y} + 3x^3y^2 + \cos(y)\right] = \frac{1}{y} + 9x^2y^2$

Hence: $\frac{\partial M}{\partial y} = \frac{1}{y} + 9x^2y^2 = \frac{\partial N}{\partial x}$

Thus, the equation IS exact, and there exists a function $U(x, y)$ such that the equation $U(x, y) = C$ relates the solution y implicitly as a function of x.

To find $U(x, y)$, we compute the integrals $\int M(x, y)\, dx$ and $\int N(x, y)\, dy$.

$U(x, y) = \int M(x, y)\, dx = \int (3x^2 + \ln(y) + 3x^2y^3)\, \partial x = x^3 + x\ln(y) + x^3y^3 + f(y)$

$U(x, y) = \int N(x, y)\, dy = \int \left(\frac{x}{y} + 3x^3y^2 + \cos(y)\right) \partial y = x\ln(y) + x^3y^3 + \sin(y) + g(x)$

To find the unknown functions $f(y)$ and $g(x)$, we compare $\int M(x, y)\, \partial x$ and $\int N(x, y)\, \partial y$.

$$\begin{array}{ccccccccccc} U(x,y) & = & x^3 & + & x\ln(y) & + & x^3y^3 & + & f(y) & + & C \\ & & \uparrow & & \uparrow & & & & \uparrow & & \\ U(x,y) & = & g(x) & + & x\ln(y) & + & x^3y^3 & + & \sin(y) & + & C \end{array}$$

Thus, $f(y) = \sin(y)$ and $g(x) = x^3$, and $U(x, y) = x^3 + x\ln(y) + x^3y^3 + \sin(y) + C$

Our solution $y = y(x)$ is given implicitly by the equation $U(x, y) = C$

$$\boxed{x^3 + x\ln(y) + x^3y^3 + \sin(y) = C_1}$$

6. Solve: $\frac{dy}{dx} = \frac{xy+y^2+x^2}{x^2}$ using the substitution $v = \frac{y}{x}$.

   1. Re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

      $\frac{dy}{dx} = \frac{xy+y^2+x^2}{x^2}$

      $\Rightarrow \frac{dy}{dx} = \frac{xy}{x^2} + \frac{y^2}{x^2} + \frac{x^2}{x^2}$

      $\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2 + 1$

   2. Substituting v for $\frac{y}{x}$, we have:

      $\frac{dy}{dx} = v + v^2 + 1$

   3. We must also make a corresponding substitution for $\frac{dy}{dx}$

      We get this from the equation $v = \frac{y}{x}$

      Rewriting, we have: $y = vx$

      $\Rightarrow \frac{d}{dx}[y] = \frac{d}{dx}[vx] \Rightarrow \frac{dy}{dx} = v \cdot 1 + x \cdot \frac{dv}{dx}$

      i.e., $\frac{dy}{dx} = v + x\frac{dv}{dx}$

      $\Rightarrow v + x\frac{dv}{dx} = v + v^2 + 1$

   4. Separate!

      $\Rightarrow x\frac{dv}{dx} = v^2 + 1$

      $\Rightarrow \frac{1}{v^2+1}dv = \frac{1}{x}dx$

      $\Rightarrow \arctan(v) = \ln(x) + C$

      $\Rightarrow \arctan\left(\frac{y}{x}\right) = \ln(x) + C$

      $\Rightarrow \frac{y}{x} = \tan(\ln(x) + C)$

      $\Rightarrow y = x\tan(\ln(x) + C)$

Our solution is: $y = x\tan(\ln(x) + C)$