MTH 3311 Test #1 – Solutions

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Show CLEARLY how you arrive at your answers.

1. Classify the following according to **order** and **linearity.** If an equation is **not linear,** explain why.

   (a) $y'' + x^2 y' = \sin(x)$ **order 2, linear.**

   The highest order of derivative of y **is** 2. (y'' is the *second derivative* of y.) Note that y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function. The equation is **linear.**

   (b) $y^{(5)} + x^2 y y'' - 2xy = 3x^2 + 2x$ **order 5, non-linear.**

   The highest order of derivative of y is 5. ($y^{(5)}$ is the *fifth derivative* of y.) Since y and y'' are cofactors in th same term, the equation is non-linear.

   (c) $e^x y''' - 3xy' + 2x^2 y = \tan(x)$ **order 3, linear.**

   The highest order of derivative of y is 3. (y''' is the *third derivative* of y.) Note that y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function. The equation is **linear.**

   (d) $y''' + 2xy'' + xyy' + xy = 6x - 6$ **order 3, non-linear.**

   The highest order of derivative of y **is** 3. (y''' is the *third derivative* of y.) Since y and y' are cofactors in th same term, the equation is non-linear.

   (e) $y^{(3)} + 6y'' + \sin(x)\, y = \frac{1}{\sqrt{9-x^2}}$ **order 3, linear.**

   The highest order of derivative of y **is** 3. ($y^{(3)}$ is the *third derivative* of y.) Note that y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function. The equation is **linear.**

2. Show that the function $\sin(2x)$ is a solution of the differential equation:

$y'' + y' + 4y = 2\cos(2x)$

Observe:

$y = \sin(2x)$

$y' = 2\cos(2x)$

$y'' = -4\sin(2x)$

Plugging these into the expression $y'' + y' + 4y$, we have:
$y'' + y' + 4y = -4\sin(2x) + 2\cos(2x) + 4\sin(2x) = 2\cos(2x)$

i.e., $y'' + y' + 4y = 2\cos(2x)$

Hence, $y = \sin(2x)$ is a solution of the equation:

$y'' + y' + 4y = 2\cos(2x)$

3. Solve: $(x^2+7)\frac{dy}{dx} = xy$; subject to the initial condition $y(3) = 12$ (Assume that $x > 0, y > 0$)

Use the "Separation of Variables" Method

$(x^2+7)\frac{dy}{dx} = xy \Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+7} \Rightarrow \frac{dy}{dx} = y\frac{x}{x^2+7} \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2+7} \Rightarrow \frac{1}{y}dy = \frac{x}{x^2+7}dx$

i.e., $\frac{1}{y}dy = \frac{x}{x^2+7}dx$

The variables are separated, now integrate!

$\int \frac{1}{y}dy = \int \frac{x}{x^2+7}dx = \int \frac{1}{x^2+7}xdx$ (Eq. 1)

$$\boxed{\begin{aligned} u &= x^2+7 \\ \frac{du}{dx} &= 2x \\ du &= 2xdx \\ \frac{1}{2}du &= xdx \end{aligned}}$$

$\Rightarrow \int \frac{1}{x^2+7}xdx = \int \underbrace{\frac{1}{x^2+7}}_{\frac{1}{u}}\underbrace{xdx}_{\frac{1}{2}du} = \int \frac{1}{u}\frac{1}{2}du = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+7| + C =$

$\ln(x^2+7)^{\frac{1}{2}} + C$

i.e., $e^y = \ln(e^x+1) + C$ (Eq. 1)

Thus, Eq. 1 becomes $\int \frac{1}{y}dy = \ln(x^2+7)^{\frac{1}{2}} + C$

i.e., $\ln(y) = \ln(x^2+7)^{\frac{1}{2}} + C$

$\Rightarrow e^{\ln(y)} = e^{\ln(x^2+7)^{\frac{1}{2}}+C} = e^{\ln(x^2+7)^{\frac{1}{2}}}e^C = C_1 e^{\ln(x^2+7)^{\frac{1}{2}}} = C_1(x^2+7)^{\frac{1}{2}} = C_1\sqrt{x^2+7}$

i.e. $y = C_1\sqrt{x^2+7}$

To find C_1, we refer to the initial condition: $y(3) = 12$

$\Rightarrow 12 = C_1\sqrt{3^2+7} = 4C_1$

i.e., $3 = C_1$

$$\boxed{\text{Our solution is } y = 3\sqrt{x^2+7}}$$

4. Solve: $\frac{1}{\cos(x)}\frac{dy}{dx} + 3y = 10$ using the "Integrating Factor" Method

1. Re-write the equation in the form: $\frac{dy}{dx} + P(x)\,y = Q(x)$

$$\frac{dy}{dx} + \underbrace{3\cos(x)}_{P(x)}y = \underbrace{10\cos(x)}_{Q(x)}$$

2. Compute the integrating factor:

$$e^{\int P(x)dx} = e^{\int 3\cos(x)dx} = e^{3\int\cos(x)dx} = e^{3\sin(x)}$$

i.e., $e^{\int P(x)dx} = e^{3\sin(x)}$

3. Multiply both sides by the integrating factor

$$e^{3\sin(x)}\frac{dy}{dx} + 3\cos(x)\,e^{3\sin(x)}y = 10\cos(x)\,e^{3\sin(x)}$$

4. Express the left hand side as the derivative of a product

$$\underbrace{e^{3\sin(x)}}_{1^{\text{st}}}\ \underbrace{\frac{dy}{dx}}_{2^{\text{nd}}\text{ prime}} + \underbrace{3\cos(x)\,e^{3\sin(x)}}_{1^{\text{st}}\text{ prime}}\ \underbrace{y}_{2^{\text{nd}}} = 10\cos(x)\,e^{3\sin(x)}$$

$$\Rightarrow \frac{d}{dx}\left[e^{3\sin(x)}y\right] = 10\cos(x)\,e^{3\sin(x)}$$

5. Integrate both sides w.r.t. x.

$$\int\frac{d}{dx}\left[e^{3\sin(x)}y\right]dx = \int 10\cos(x)\,e^{3\sin(x)}dx$$

$$\Rightarrow e^{3\sin(x)}y = 10\int\underbrace{e^{3\sin(x)}}_{e^u}\underbrace{\cos(x)\,dx}_{\frac{1}{3}du} = \frac{10}{3}e^{3\sin(x)} + C$$

i.e., $e^{3\sin(x)}y = \frac{10}{3}e^{3\sin(x)} + C$

6. Solve for y

$$\Rightarrow y = \frac{10}{3} + Ce^{-3\sin(x)}$$

Our solution is $y = \frac{10}{3} + Ce^{-3\sin(x)}$

5. Determine whether or not the equation is exact. If the equation is exact, solve it.
$(8\cos(x) + 12x^2y^2)\,dx + \left(8x^3y + \frac{6}{y}\right)dy = 0$

This equation can be analyzed as:

$$\underbrace{\left(8\cos(x) + 12x^2y^2\right)}_{M(x,y)}dx + \underbrace{\left(8x^3y + \frac{6}{y}\right)}_{N(x,y)}dy = 0$$

By convention, we let $M(x,y)$ be the co-factor of dx and we let $N(x,y)$ be the co-factor of dy.

i.e., $M(x,y) = 8\cos(x) + 12x^2y^2$ and $N(x,y) = 8x^3y + \frac{6}{y}$

If the Differential equation is **exact,** then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = 24x^2y = \frac{\partial N}{\partial x}$

Thus, the equation IS exact, and there exists a function $U(x,y)$ such that the equation $U(x,y) = C$ relates the solution y implicitly as a function of x.

To find $U(x,y)$, we compute the integrals $\int M(x,y)\,dx$ and $\int N(x,y)\,dy$.

$U(x,y) = \int M(x,y)\,dx = \int (8\cos(x) + 12x^2y^2)\,\partial x = 8\sin(x) + 4x^3y^2 + f(y)$

$U(x,y) = \int N(x,y)\,dy = \int \left(8x^3y + \frac{6}{y}\right)\partial y = 4x^3y^2 + 6\ln|y| + g(x)$

To find the unknown functions $f(y)$ and $g(x)$, we compare $\int M(x,y)\,\partial x$ and $\int N(x,y)\,\partial y$.

$$\begin{array}{ccccccccc} U(x,y) & = & 8\sin(x) & + & 4x^3y^2 & + & f(y) & + & C \\ & & \uparrow & & \uparrow & & \uparrow & & \\ U(x,y) & = & g(x) & + & 4x^3y^2 & + & 6\ln|y| & + & C \end{array}$$

Thus, $f(y) = 6\ln|y|$ and $g(x) = 8\sin(x)$, and $U(x,y) = 8\sin(x) + 4x^3y^2 + 6\ln|y| + C$

Our solution $y = y(x)$ is given implicitly by the equation $U(x,y) = C$

$$\boxed{8\sin(x) + 4x^3y^2 + 6\ln|y| = C_1}$$

6. Solve: $x^2\frac{dy}{dx} = x^2 + xy + y^2$ using the substitution $v = \frac{y}{x}$. (Assume that $x, y > 0$)

1. Re-express this in the form: $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$

$x^2\frac{dy}{dx} = x^2 + xy + y^2$

$\Rightarrow \frac{dy}{dx} = \frac{x^2}{x^2} + \frac{xy}{x^2} + \frac{y^2}{x^2}$

$\Rightarrow \frac{dy}{dx} = 1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2$

2. Substituting v for $\frac{y}{x}$, we have:

$\frac{dy}{dx} = 1 + v + v^2$

3. We must also make a corresponding substitution for $\frac{dy}{dx}$

We get this from the equation $v = \frac{y}{x}$

Rewriting, we have: $y = vx$

$\Rightarrow \frac{d}{dx}[y] = \frac{d}{dx}[vx] \Rightarrow \frac{dy}{dx} = v \cdot 1 + x \cdot \frac{dv}{dx}$

i.e., $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow v + x\frac{dv}{dx} = 1 + v + v^2$

4. Separate!

$\Rightarrow x\frac{dv}{dx} = 1 + v^2$

$\Rightarrow \frac{1}{1+v^2}dv = \frac{1}{x}dx$

$\Rightarrow \arctan(v) = \ln(x) + C$

$\Rightarrow \arctan\left(\frac{y}{x}\right) = \ln(x) + C$

$\Rightarrow \frac{y}{x} = \tan(\ln(x) + C)$

$\Rightarrow y = x\tan(\ln(x) + C)$

Our solution is: $y = x\tan(\ln(x) + C)$