MTH 1125 Test #1 - Solutions

Summer 2014

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\lim_{x\to 1} \frac{x^2+2x+4}{x^3-2x+4} =$

   (a) 1. Try Plugging in:

   $\lim_{x\to 1} \frac{x^2+2x+4}{x^3-2x+4} = \frac{(1)^2+2(1)+4}{(1)^3-2(1)+4} = \frac{7}{3}$

   $\boxed{\text{i.e., } \lim_{x\to 1} \frac{x^2+2x+4}{x^3-2x+4} = \frac{7}{3}}$

2. Compute: $\lim_{x\to 3} \frac{x^2-4x+3}{x^2-x-6} =$

   (a) 1. Try Plugging in:

   $\lim_{x\to 3} \frac{x^2-4x+3}{x^2-x-6} = \frac{(3)^2-4(3)+3}{(3)^2-(3)-6} = \frac{0}{0}$ No Good - Zero Divide!

   2. Try Factoring and Cancelling:

   $\lim_{x\to 3} \frac{x^2-4x+3}{x^2-x-6} = \lim_{x\to 3} \frac{(x-1)(x-3)}{(x+2)(x-3)} = \lim_{x\to 3} \frac{(x-1)}{(x+2)} = \frac{(3)-1}{(3)+2} = \frac{2}{5}$

   $\boxed{\text{i.e., } \lim_{x\to 3} \frac{x^2-4x+3}{x^2-x-6} = \frac{2}{5}}$

3. Compute: $\lim_{x\to -\infty} \frac{10x^3+4x^2+2}{5x^4-8x^3+2x^2+x} =$

   $\lim_{x\to -\infty} \frac{10x^3+4x^2+2}{5x^4-8x^3+2x^2+x} = \lim_{x\to -\infty} \frac{10x^3}{5x^4} = \lim_{x\to -\infty} \frac{2}{x} = 0$

   $\boxed{\text{i.e., } \lim_{x\to -\infty} \frac{10x^3+4x^2+2}{5x^4-8x^3+2x^2+x} = 0}$

4. Compute: $\lim_{x\to -2} \frac{x-1}{x^2-x-6} =$

(a) 1. Try Plugging in:

$\lim_{x\to -2} \frac{x-1}{x^2-x-6} = \frac{(-2)-1}{(-2)^2-(-2)-6} = \frac{-3}{0}$ No Good - Zero Divide!

2. Try Factoring and Cancelling:

No Good!. "Factoring and Cancelling" only works when Step #1 yields $\frac{0}{0}$.

3. Analyze the one-sided limits:

$\lim_{x\to -2^-} \frac{x-1}{x^2-x-6} = \lim_{x\to -2^-} \frac{x-1}{(x+2)(x-3)} = \frac{-3}{(-\varepsilon)(-5)} = \frac{\left(\frac{-3}{-5}\right)}{(-\varepsilon)} = -\infty$

$$\boxed{\begin{array}{ll} & x \to -2^- \\ \Rightarrow & x < -2 \\ \Rightarrow & x+2 < 0 \end{array}}$$

$\lim_{x\to -2^+} \frac{x-1}{x^2-x-6} = \lim_{x\to -2^+} \frac{x-1}{(x+2)(x-3)} = \frac{-3}{(+\varepsilon)(-5)} = \frac{\left(\frac{-3}{-5}\right)}{(+\varepsilon)} = \frac{\left(\frac{3}{5}\right)}{(+\varepsilon)} = +\infty$

$$\boxed{\begin{array}{ll} & x \to -2^+ \\ \Rightarrow & x > -2 \\ \Rightarrow & x+2 > 0 \end{array}}$$

Since the one-sided limits are not equal, $\lim_{x\to -2} \frac{x-1}{x^2-x-6}$ **Does Not Exist!**

5. $f(x) = 8x^4 + 10x^3 + 16x^2 + 30x + 8\sqrt{x} + 4$; Compute: $f'(x)$.

Rewrite: $f(x) = 8x^4 + 10x^3 + 16x^2 + 30x + 8x^{\frac{1}{2}} + 4$

$f'(x) = 8(4x^3) + 10(3x^2) + 16(2x^1) + 30(1) + 8\left(\frac{1}{2}x^{-\frac{1}{2}}\right) + 0 = 32x^3 + 30x^2 + 32x + 30 + 4x^{-\frac{1}{2}}$

i.e., $f'(x) = 32x^3 + 30x^2 + 32x + 30 + 4x^{-\frac{1}{2}}$

6. $\frac{d}{dx}\left[-3\sin(x)+6\cos(x)\right]=$

$\frac{d}{dx}\left[-3\sin(x)+6\cos(x)\right]=-3\left(\cos(x)\right)+6\left(-\sin(x)\right)=-3\cos(x)-6\sin(x)$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[-3\sin(x)+6\cos(x)\right]=-3\cos(x)-6\sin(x)}$$

7. Find the asymptotes and graph: $f(x)=\frac{x-1}{x^2+x-6}$

$\boxed{\text{Verticals}}$

1. Find x-values that cause division by zero.

$\Rightarrow x^2+x-6=0$

$\Rightarrow (x+3)(x-2)=0$

$\Rightarrow x=-3$ and $x=2$ are possible vertical asymptotes.

2. Compute the one-sided limits.

$\lim_{x\to -3^-}\frac{x-1}{x^2+x-6}=\lim_{x\to -3^-}\frac{x-1}{(x+3)(x-2)}=\frac{-4}{(-\varepsilon)(-5)}=\frac{\left(\frac{-4}{-5}\right)}{(-\varepsilon)}=\frac{\left(\frac{4}{5}\right)}{-\varepsilon}=-\infty$

$$\boxed{\begin{array}{ll} & x\to -3^- \\ \Rightarrow & x<-3 \\ \Rightarrow & x+3<0 \end{array}}$$

$\lim_{x\to -3^+}\frac{x-1}{x^2+x-6}=\lim_{x\to -3^+}\frac{x-1}{(x+3)(x-2)}=\frac{-4}{(\varepsilon)(-5)}=\frac{\left(\frac{-4}{-5}\right)}{(\varepsilon)}=\frac{\left(\frac{4}{5}\right)}{\varepsilon}=+\infty$

$$\boxed{\begin{array}{ll} & x\to -3^+ \\ \Rightarrow & x>-3 \\ \Rightarrow & x+3>0 \end{array}}$$

Since the one-sided limits are infinite, $x=-3$ *IS a vertical asymptote.*

$\lim_{x\to 2^-}\frac{x-1}{x^2+x-6}=\lim_{x\to 2^-}\frac{x-1}{(x+3)(x-2)}=\frac{1}{(5)(-\varepsilon)}=\frac{\left(\frac{1}{5}\right)}{(-\varepsilon)}=-\infty$

$$\boxed{\begin{array}{ll} & x\to 2^- \\ \Rightarrow & x<2 \\ \Rightarrow & x-2<0 \end{array}}$$

$\lim_{x\to 2^+}\frac{x-1}{x^2+x-6}=\lim_{x\to 2^+}\frac{x-1}{(x+3)(x-2)}=\frac{1}{(5)(+\varepsilon)}=\frac{\left(\frac{1}{5}\right)}{(+\varepsilon)}=+\infty$

$$\boxed{\begin{array}{ll} & x\to 2^+ \\ \Rightarrow & x>2 \\ \Rightarrow & x-2>0 \end{array}}$$

Since the one-sided limits are infinite, $x=2$ *IS a vertical asymptote.*

$\boxed{\text{Horizontals}}$

Compute the limits as $x \to -\infty$ and as $x \to +\infty$

$\lim_{x\to-\infty} \frac{x-1}{x^2+x-6} = \lim_{x\to-\infty} \frac{x}{x^2} = \lim_{x\to-\infty} \frac{1}{x} = 0$

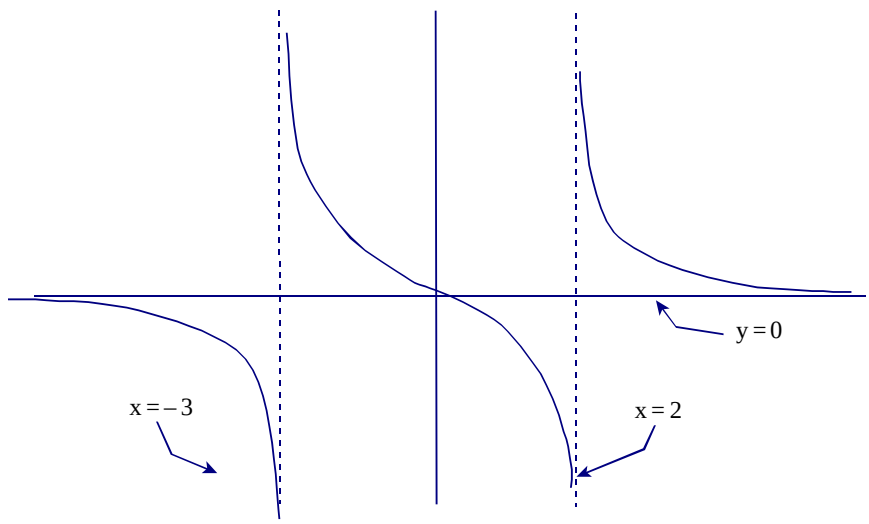
$\lim_{x\to+\infty} \frac{x-1}{x^2+x-6} = \lim_{x\to+\infty} \frac{x}{x^2} = \lim_{x\to+\infty} \frac{1}{x} = 0$

Since the limits are finite and constant, $y = 0$ is a horizontal asymptote.

Summary:

$\lim_{x\to-3^-} \frac{x-1}{x^2+x-6} = -\infty$	
$\lim_{x\to-3^+} \frac{x-1}{x^2+x-6} = +\infty$	$\lim_{x\to-\infty} \frac{x-1}{x^2+x-6} = 0$
$\lim_{x\to2^-} \frac{x-1}{x^2+x-6} = -\infty$	$\lim_{x\to+\infty} \frac{x-1}{x^2+x-6} = 0$
$\lim_{x\to2^+} \frac{x-1}{x^2+x-6} = +\infty$	

Graph $f(x) = \frac{x-1}{x^2+x-6}$



8.

$x =$	$f(x) =$
3.5	-15.1
3.9	-227.8
3.99	-1212.3
3.999	-21156.3
3.9999	-834561.9

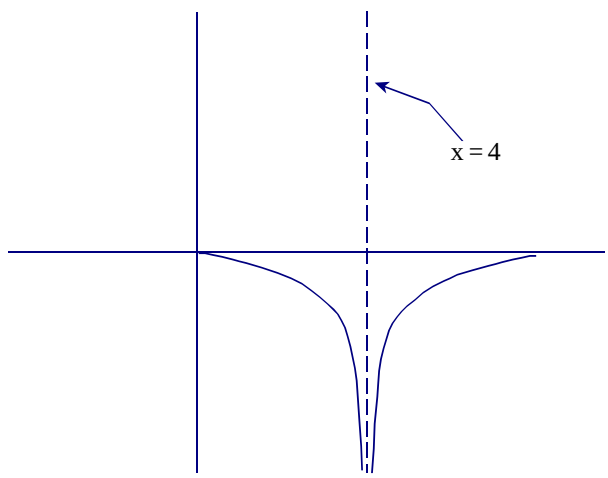
$x =$	$f(x) =$
4.5	-15.1
4.1	-227.8
4.01	-1212.3
4.001	-21156.3
4.0001	-834561.9

Based on the information in the table above, do the following:

(a) $\lim_{x \to 4^-} f(x) = -\infty$

(b) $\lim_{x \to 4^+} f(x) = -\infty$

(c) Graph $f(x)$



9. Compute: $\lim_{x\to 5} \frac{\sqrt{20+x}-5}{x-5} =$

(a) 1. Try Plugging in:

$\lim_{x\to 5} \frac{\sqrt{20+x}-5}{x-5} = \frac{\sqrt{20+(5)}-5}{(5)-5} = \frac{0}{0}$ No Good - Zero Divide!

2. Try Factoring and Cancelling:

$\lim_{x\to 5} \frac{\sqrt{20+x}-5}{x-5} = \lim_{x\to 5} \frac{\sqrt{20+x}-5}{x-5} \cdot \frac{\sqrt{20+x}+5}{\sqrt{20+x}+5} = \lim_{x\to 5} \frac{\left(\sqrt{20+x}\right)^2-(5)^2}{(x-5)\left[\sqrt{20+x}+5\right]}$

$= \lim_{x\to 5} \frac{20+x-25}{(x-5)\left[\sqrt{20+x}+5\right]} = \lim_{x\to 5} \frac{(x-5)}{(x-5)\left[\sqrt{20+x}+5\right]} = \lim_{x\to 5} \frac{1}{\sqrt{20+x}+5} = \frac{1}{\sqrt{20+(5)}+5} =$

$\frac{1}{5+5} = \frac{1}{10}$

$\boxed{\text{i.e., } \lim_{x\to 5} \frac{\sqrt{20+x}-5}{x-5} = \frac{1}{10}}$

10. $f(x) = 3x^2 + 5x + 1$; compute $f'(x)$ **using the definition of derivative.** (i.e., compute $f'(x)$ using the "limiting process.")

$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\left[3(x+\Delta x)^2+5(x+\Delta x)+1\right]-\left(3x^2+5x+1\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\left[3\left(x^2+2x\Delta x+\Delta x^2\right)+5(x+\Delta x)+1\right]-\left(3x^2+5x+1\right)}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\left[\left(3x^2+6x\Delta x+3\Delta x^2\right)+(5x+5\Delta x)+1\right]-\left(3x^2+5x+1\right)}{\Delta x} = \lim_{\Delta x\to 0} \frac{6x\Delta x+3\Delta x^2+5\Delta x}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{\Delta x(6x+3\Delta x+5)}{\Delta x} = \lim_{\Delta x\to 0} (6x + 3\Delta x + 5) = 6x + 3(0) + 5 = 6x + 5$

$\boxed{\text{i.e., } f'(x) = 6x + 5}$

11. Compute: $\frac{d}{dx}\left[x^2\tan(x)\right] =$

$$\frac{d}{dx}\left[\underbrace{x^2}_{1^{\text{st}}}\cdot\underbrace{\tan(x)}_{2^{\text{nd}}}\right] = \underbrace{2x}_{1^{\text{st}}\text{ prime}}\cdot\underbrace{\tan(x)}_{2^{\text{nd}}} + \underbrace{\sec^2(x)}_{2^{\text{nd}}\text{prime}}\cdot\underbrace{x^2}_{1^{\text{st}}} = 2x\tan(x) + x^2\sec^2(x)$$

$$\boxed{\text{i.e., } \frac{d}{dx}\left[x^2\tan(x)\right] = 2x\tan(x) + x^2\sec^2(x)}$$

12. Compute: $\frac{d}{dx}\left[\frac{\sec(x)}{x^2+1}\right] =$

$$\frac{d}{dx}\left[\frac{\overbrace{\sec(x)}^{\text{top}}}{\underbrace{x^2+1}_{\text{bottom}}}\right] = \frac{\overbrace{\sec(x)\tan(x)}^{\text{top prime}}\cdot\overbrace{\left(x^2+1\right)}^{\text{bottom}} - \overbrace{2x}^{\text{bottom prime}}\cdot\overbrace{\sec(x)}^{\text{top}}}{\underbrace{\left(x^2+1\right)^2}_{\text{bottom squared}}}$$

$$\boxed{\text{i.e., } f'(x) = \frac{\sec(x)\tan(x)\left(x^2+1\right)-2x\sec(x)}{\left(x^2+1\right)^2}}$$

EXTRA - WOW! (7 pts) Determine whether or not $f(x)$ is continuous at the point $x = 3$. (Justify your answer.)

13. $f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{for } x < 3 \\ 2 & \text{for } x = 3 \\ 12-2x & \text{for } x > 3 \end{cases}$

If $f(x)$ is continuous at the point $x = 3$, then $\lim_{x\to 3} f(x) = f(3)$.

To see if this is true, we'll compute $\lim_{x\to 3} f(x)$.

Since the definiton of $f(x)$ for $x < 3$ is different than the definiton of $f(x)$ for $x > 3$, we must compute the one-sided limits in order to determine whether the limit exists.

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x^2-9}{x-3} = \lim_{x\to 3^-} \frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3^-}(x+3) = 6$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(12-2x) = 12-2(3) = 6$

Since the one-sided limits ARE EQUAL, $\lim_{x\to 3} f(x)$ EXISTS and is equal to the common value of the one-sided limits.

i.e., $\lim_{x\to 3} f(x) = 6$

However, $f(3) = 2$

Therefore, since $\lim_{x\to 3} f(x) \neq f(3)$,

$f(x)$ is NOT continuous at $x = 3$