MTH 1125 12pm Class - Test #4 - Solutions

FALL 2019

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Show CLEARLY how you arrive at your answers!

1. **Compute:** $\int \left(8x^3 + 6x^2 + 4x + 2 + 2\sqrt{x}\right) dx =$

$\int \left(8x^3 + 6x^2 + 4x + 2 + 2\sqrt{x}\right) dx = \int \left(8x^3 + 6x^2 + 4x + 2 + 2x^{\frac{1}{2}}\right) dx$

$= 8\left[\frac{x^4}{4}\right] + 6\left[\frac{x^3}{3}\right] + 4\left[\frac{x^2}{2}\right] + 2x + 2\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + C = 2x^4 + 2x^3 + 2x^2 + 2x + 2\left(\frac{2}{3}\right)x^{\frac{3}{2}} + C$

$= 2x^4 + 2x^3 + 2x^2 + 2x + \frac{4}{3}x^{\frac{3}{2}} + C$

i.e., $\int \left(8x^3 + 6x^2 + 4x + 2 + 2\sqrt{x}\right) dx = 2x^4 + 2x^3 + 2x^2 + 2x + \frac{4}{3}x^{\frac{3}{2}} + C$

2. **Compute:** $\int \left(4x^3+12x+2\right)^9 \left(x^2+1\right) dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(4x^3+12x+2\right)^9$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(4x^3+12x+2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(4x^3+12x+2\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(x^2+1\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(4x^3+12x+2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 4x^3+12x+2 \\ \Rightarrow \tfrac{du}{dx} &= 12x^2+12 \\ \Rightarrow du &= \left(12x^2+12\right) dx \\ \Rightarrow \tfrac{1}{12} du &= \left(x^2+1\right) dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(4x^3+12x+2\right)^9}_{u^9} \underbrace{\left(x^2+1\right) dx}_{\frac{1}{12}du} = \int u^9 \tfrac{1}{12} du = \tfrac{1}{12} \int u^9 du$$

4. Integrate (in terms of u).

$$\tfrac{1}{12} \int u^9 du = \tfrac{1}{12} \left[\tfrac{u^{10}}{10}\right] + C = \tfrac{1}{120} u^{10} + C$$

5. Re-express in terms of the original variable, x.

$$\int \left(4x^3+12x+2\right)^9 \left(x^2+1\right) dx = \underbrace{\frac{1}{120} \left(4x^3+12x+2\right)^{10} + C}_{\frac{1}{120}u^{10}+C}$$

i.e., $\int \left(4x^3+12x+2\right)^9 \left(x^2+1\right) dx = \frac{1}{120} \left(4x^3+12x+2\right)^{10} + C$

3. **Compute:** $\int \left(5\cos(x) + \csc^2(x) - 6\sec(x)\tan(x)\right) dx =$

$\int \left(5\cos(x) + \csc^2(x) - 6\sec(x)\tan(x)\right) dx = 5\left[\sin(x)\right] + \left[-\cot(x)\right] - 6\left[\sec(x)\right] + C$

$= 5\sin(x) - \cot(x) - 6\sec(x) + C$

i.e., $\int \left(5\cos(x) + \csc^2(x) - 6\sec(x)\tan(x)\right) dx = 5\sin(x) - \cot(x) - 6\sec(x) + C$

4. **Compute:** $\int \sin\left(9x^2+16x+4\right)(9x+8)\,dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\quad \sin\left(9x^2+16x+4\right)$

   $\nearrow \qquad\qquad \nwarrow$

   outer $\qquad\qquad\qquad$ inner

   Let u = the "inner" of the composite function

   $\Rightarrow u = \left(9x^2+16x+4\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\quad \underbrace{\left(9x^2+16x+4\right)}_{\text{function}} - - - - \rightarrow \underbrace{(9x+8)}_{\text{deriv}}$

   Let u = the "function" of the function/deriv pair

   $\Rightarrow u = \left(9x^2+16x+4\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 9x^2+16x+4 \\ \Rightarrow \frac{du}{dx} &= 18x+16 \\ \Rightarrow du &= (18x+16)\,dx \\ \Rightarrow \tfrac{1}{2}du &= (9x+8)\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\sin\left(9x^2+16x+4\right)}_{\sin(u)}\underbrace{(9x+8)\,dx}_{\frac{1}{2}du} = \int \sin(u)\ \tfrac{1}{2}du = \tfrac{1}{2}\int \sin(u)\,du$$

4. Integrate (in terms of u).

$$\tfrac{1}{2}\int \sin(u)\,du = \tfrac{1}{2}\left[-\cos(u)\right] + C = -\tfrac{1}{2}\cos(u) + C$$

5. Re-express in terms of the original variable, x.

$$\int \sin\left(9x^2+16x+4\right)(9x+8)\,dx = \underbrace{-\frac{1}{2}\cos\left(9x^2+16x+4\right)+C}_{-\frac{1}{2}\cos(u)+C}$$

i.e., $\int \sin\left(6x^2+18x+4\right)(9x+8)\,dx = -\frac{1}{2}\cos\left(9x^2+16x+4\right)+C$

5. **Compute:** $\int_0^1 \left(8x^3 + 6x^2 + 2\right) dx =$

$$\int_0^1 \underbrace{\left(8x^3 + 6x^2 + 2\right)}_{f(x)} dx = \underbrace{\left[8\frac{x^4}{4} + 6\frac{x^3}{3} + 2x\right]_0^1}_{F(x)} = \underbrace{\left[2x^4 + 2x^3 + 2x\right]_0^1}_{F(x)} =$$

$$= \underbrace{\left[2\,(1)^4 + 2\,(1)^3 + 2\,(1)\right]}_{F(1)} - \underbrace{\left[2\,(0)^4 + 2\,(0)^3 + 2\,(0)\right]}_{F(0)} = 6$$

i.e., $\int_0^1 \left(8x^3 + 6x^2 + 2\right) dx = 6$

6. **Compute:** $\int_0^1 \left(4x^3+2\right)^3 x^2\ dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(4x^3+2\right)^3$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(4x^3+2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(4x^3+2\right)}_{\text{function}} - - - - \rightarrow \underbrace{x^2}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(4x^3+2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 4x^3+2 \\ \Rightarrow \tfrac{du}{dx} &= 12x^2 \\ \Rightarrow du &= 12x^2 dx \\ \Rightarrow \tfrac{1}{12} du &= x^2\ dx \end{aligned}$$

When $x = 0$, $u = 4x^3+2 = 4\left(0\right)^3+2 = 2$
When $x = 1$, $u = 4x^3+2 = 4\left(1\right)^3+2 = 6$

3. Analyze in terms of u and du

$$\int_{x=0}^{x=1} \underbrace{\left(4x^3+2\right)^3}_{u^3} \underbrace{x^2\ dx}_{\frac{1}{12}du} = \int_{u=2}^{u=6} u^3 \cdot \tfrac{1}{12} du = \tfrac{1}{12}\int_{u=2}^{u=6} u^3 du$$

Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$\tfrac{1}{12}\int_{u=2}^{u=6} u^3 du = \tfrac{1}{12}\left[\tfrac{u^4}{4}\right]_{u=2}^{u=6} = \left[\tfrac{u^4}{48}\right]_{u=2}^{u=6} = \underbrace{\frac{\left(6\right)^4}{48}}_{F(6)} - \underbrace{\frac{\left(2\right)^4}{48}}_{F(2)} = \tfrac{1296}{48} - \tfrac{16}{48} = \tfrac{1280}{48} = \tfrac{80}{3}$$

i.e., $\int_{x=0}^{x=1} \left(4x^3+2\right)^3 x^2\ dx = \frac{80}{3}$

7. Compute: $\int \frac{3\cos(x)-2\sin(x)}{3\sin(x)+2\cos(x)}dx =$

$$\int \frac{3\cos(x)-2\sin(x)}{3\sin(x)+2\cos(x)}dx \underbrace{=}_{\text{re-write}} \int \frac{1}{3\sin(x)+2\cos(x)} \left(3\cos(x) - 2\sin(x)\right) dx$$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{3\sin(x)+2\cos(x)}$ is the same as $\left(3\sin(x) + 2\cos(x)\right)^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = 3\sin(x) + 2\cos(x)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{(3\sin(x) + 2\cos(x))}_{\text{function}} - - - - \rightarrow \underbrace{(3\cos(x) - 2\sin(x))}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = 3\sin(x) + 2\cos(x)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 3\sin(x) + 2\cos(x) \\ \Rightarrow \frac{du}{dx} &= 3\cos(x) - 2\sin(x) \\ \Rightarrow du &= (3\cos(x) - 2\sin(x))\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\underbrace{\int \frac{1}{3\sin(x) + 2\cos(x)}}_{\frac{1}{u}} \underbrace{(3\cos(x) - 2\sin(x))\ dx}_{du} = \int \frac{1}{u} du$$

4. Integrate (in terms of u).

   $\int \frac{1}{u} du = \ln|u| + C$

5. Re-express in terms of the original variable, x.

   $\int \frac{3\cos(x)-2\sin(x)}{3\sin(x)+2\cos(x)}dx = \underbrace{\ln|3\sin(x) + 2\cos(x)| + C}_{\ln|u|+C}$

   i.e., $\int \frac{3\cos(x)-2\sin(x)}{3\sin(x)+2\cos(x)}dx = \ln|3\sin(x) + 2\cos(x)| + C$

8. **Compute:** $\int \frac{3\cos(x)-2\sin(x)}{\sqrt{3\sin(x)+2\cos(x)}}dx =$

$$\int \frac{3\cos(x)-2\sin(x)}{\sqrt{3\sin(x)+2\cos(x)}}dx \underbrace{=}_{\text{re-write}} \int \frac{1}{(3\sin(x)+2\cos(x))^{\frac{1}{2}}} \left(3\cos(x)-2\sin(x)\right) dx \underbrace{=}_{\text{re-write}} \int \left(3\sin(x)+2\cos(x)\right)^{-\frac{1}{2}} \left(3\cos(x)-2\sin\right.$$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(3\sin(x)+2\cos(x)\right)^{-\frac{1}{2}}$ is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = 3\sin(x)+2\cos(x)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(3\sin(x)+2\cos(x)\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(3\cos(x)-2\sin(x)\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = 3\sin(x)+2\cos(x)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{array}{rcl} u & = & 3\sin(x)+2\cos(x) \\ \Rightarrow \frac{du}{dx} & = & 3\cos(x)-2\sin(x) \\ \Rightarrow du & = & \left(3\cos(x)-2\sin(x)\right) dx \end{array}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(3\sin(x)+2\cos(x)\right)^{-\frac{1}{2}}}_{u^{-\frac{1}{2}}} \underbrace{\left(3\cos(x)-2\sin(x)\right) dx}_{du} = \int u^{-\frac{1}{2}} du$$

4. Integrate (in terms of u).

$$\int u^{-\frac{1}{2}} du = \frac{u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + C = 2u^{\frac{1}{2}} + C$$

5. Re-express in terms of the original variable, x.

$$\int \frac{3\cos(x)-2\sin(x)}{\sqrt{3\sin(x)+2\cos(x)}}dx = \underbrace{2\left(3\sin(x)+2\cos(x)\right)^{\frac{1}{2}} + C}_{2u^{\frac{1}{2}}+C}$$

i.e., $\int \frac{3\cos(x)-2\sin(x)}{\sqrt{3\sin(x)+2\cos(x)}}dx = 2\left(3\sin(x)+2\cos(x)\right)^{\frac{1}{2}} + C$

9. Compute: $\frac{d}{dx}\left[\ln\left(\sec(x)+\tan(x)\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(\sec(x)+\tan(x)\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{\sec(x)+\tan(x)}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(\sec(x)\tan(x)+\sec^2(x)\right)}_{g'(x)} = \frac{\sec(x)\tan(x)+\sec^2(x)}{\sec(x)+\tan(x)}$$

$$= \frac{\sec(x)(\tan(x)+\sec(x))}{\sec(x)+\tan(x)} = \sec(x)$$

i.e., $\frac{d}{dx}\left[\ln\left(\sec(x)+\tan(x)\right)\right] = \frac{\sec(x)\tan(x)+\sec^2(x)}{\sec(x)+\tan(x)} = \sec(x)$

10. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2+1}{2x^3}}\right)\right] =$ (Use the algebraic properties of natural logarithms to simplify first)

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2+1}{2x^3}}\right)\right] \underbrace{=}_{\text{re-write}} \frac{d}{dx}\left[\ln\left[\left(\frac{x^2+1}{2x^3}\right)^{\frac{1}{2}}\right]\right] \underbrace{=}_{\text{re-write}} \frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{x^2+1}{2x^3}\right)\right] \underbrace{=}_{\text{re-write}} \frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(x^2+1\right)-\ln\left(2x^3\right)\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{x^2+1}(2x) - \frac{1}{2x^3}\left(6x^2\right)\right) = \left(\frac{1}{2}\frac{1}{x^2+1}(2x) + \frac{1}{2}\frac{1}{2x^3}\left(6x^2\right)\right) = \left(\frac{x}{x^2+1} + \frac{3}{2x}\right)$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2+1}{2x^3}}\right)\right] = \left(\frac{x}{x^2+1} + \frac{3}{2x}\right)$