MTH 3311 - Test #3 - Solutions

Fall 2018

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Show CLEARLY how you arrive at your answers!

1. Find the general solution of the differential equation: $y'' + y' = \cos(x)$

   First, find the solution to the complementary equation $y'' + y' = 0$

   The auxiliary equation is $m^2 + m = 0$

   $\Rightarrow m(m+1) = 0 \Rightarrow m_1 = 0$ and $m_2 = -1$

   This yields: $y_c = c_1 e^{0x} + c_2 e^{-1x} = c_1 e^0 + c_2 e^{-x}$

   i.e., $y_c = c_1 + c_2 e^{-x}$

   For the particular solution, we consider the original equation: $y'' + y' = \cos(x)$

   Since the right hand side of the equation is a linear combination of sines and cosines, we imagine that the particular solution is also a linear combination of sines and cosines:

   $y_p = A\cos(x) + B\sin(x)$

   $\Rightarrow y_p' = -A\sin(x) + B\cos(x)$

   $\Rightarrow y_p'' = -A\cos(x) - B\sin(x)$

   Next, we plug these expressions for y, y', y'' into the equation $y'' + y' = \cos(x)$, and solve for A and B.

   This yields:

   $$\underbrace{-A\cos(x) - B\sin(x)}_{y''} + \underbrace{-A\sin(x) + B\cos(x)}_{y'} = \cos(x)$$

   $\Rightarrow (-A - B)\sin(x) + (-A + B)\cos(x) = 0\sin(x) + \cos(x)$

   Comparing the coefficients of the different powers of $\sin(x)$ and $\cos(x)$, we get:

   $$\begin{array}{rrrrrrl} - & A & - & B & = & 0 & \text{(Eq. 1)} \\ - & A & + & B & = & 1 & \text{(Eq. 2)} \\ \hline - & 2A & & & = & 1 & \text{(Eq. 3)} \quad \text{(The sum of (Eq. 1) + (Eq. 2)} \end{array}$$

   From Eq. 3, we get: $A = -\frac{1}{2}$

Plugging $A = -\frac{1}{2}$ into Eq. 2, we get: $B = \frac{1}{2}$

Hence, $y_p = -\frac{1}{2}\cos(x) + \frac{1}{2}\sin(x)$

The solution to the original equation is: $y = y_p + y_c$

$$\boxed{\Rightarrow y = -\frac{1}{2}\cos(x) + \frac{1}{2}\sin(x) + c_1 + c_2 e^{-x}}$$

2. Find the general solution of the differential equation: $y'' - 9y = e^{3x}$

First, find the solution to the complementary equation $y'' - 9y = 0$

The auxiliary equation is $m^2 - 9 = 0$

$\Rightarrow m^2 - 9 = 0 \Rightarrow (m+3)(m-3) = 0 \Rightarrow m_1 = -3$ and $m_2 = 3$

This yields: $y_c = c_1 e^{-3x} + c_2 e^{3x}$

For the particular solution, we consider the original equation: $y'' - 9y = e^{3x}$

Since the right hand side of the equation is an exponential function, we imagine that the particular solution is also an exponential function:

$y_p = Ae^{3x}$

Oops! This is one of the independent solutions of the *complementary* solution.

To remedy this situation, we multiply by x.

$\Rightarrow y_p = Axe^{3x}$

$\Rightarrow y_p' = Ae^{3x} + 3Axe^{3x}$

$\Rightarrow y_p'' = 3Ae^{3x} + 3Ae^{3x} + 9Axe^{3x}$

i.e., $y_p'' = 6Ae^{3x} + 9Axe^{3x}$

To find A, we plug these into the original equation, $y'' - 9y = e^{3x}$.

This yields:

$$\underbrace{6Ae^{3x} + 9Axe^{3x}}_{y''} - \underbrace{9\left(Axe^{3x}\right)}_{9y} = e^{3x}$$

$\Rightarrow 6Ae^{3x} = e^{3x}$

$\Rightarrow 6A = 1$

$\Rightarrow A = \frac{1}{6}$

Hence, $y_p = \frac{1}{6}xe^{3x}$

The solution to the original equation is: $y = y_p + y_c$

$$\boxed{\Rightarrow y = \frac{1}{6}xe^{3x} + c_1 e^{-3x} + c_2 e^{3x}}$$

3. Find the general solution of the differential equation: $y'' + y = \csc^2 x$

First, find the solution to the complementary equation $y'' + y = 0$

The auxiliary equation is $m^2 + 1 = 0$

$\Rightarrow m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i \Rightarrow m_1 = i$ and $m_2 = -i$

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{ix} + c_2 e^{-ix} = A\cos(x) + B\sin(x)$

i.e., $y_c = A\cos(x) + B\sin(x)$

Next, we find the particular solution.

Since the right hand side of the original equation is not a linear combination of sines, cosines, exponentials, and polynomials, the method of Undetermined Coefficients won't work.

Therefore we must use Variation of Parameters.

We convert the complementary solution into the general solution by transforming the parameters A and B into the functions $A(x)$ and $B(x)$, yielding:

$y = A(x)\cos(x) + B(x)\sin(x)$

We have two restrictions that we can impose on this pair of functions.

The first restriction that we impose is that the pair $A(x), B(x)$ actually does make the equation

$y = A(x)\cos(x) + B(x)\sin(x)$ the general solution.

$y = A(x)\cos(x) + B(x)\sin(x)$

$\Rightarrow y' = A(x)(-\sin(x)) + A'(x)\cos(x) + B(x)\cos(x) + B'(x)\sin(x)$

To simplify this expression, we impose our **second restriction:**

$A'(x)\cos(x) + B'(x)\sin(x) = 0 \quad \text{(Eq.1)}$

$\Rightarrow y' = -A(x)\sin(x) + B(x)\cos(x)$

$\Rightarrow y'' = -A(x)\cos(x) - A'(x)\sin(x) + B(x)(-\sin(x)) + B'(x)\cos(x)$

i.e., $y'' = -A(x)\cos(x) - A'(x)\sin(x) - B(x)\sin(x) + B'(x)\cos(x)$

We plug these into the original equation: $y'' + y = \csc^2(x)$

This yields:

$$\begin{array}{rrrrrrl} & y'' = & -A(x)\cos(x) & -A'(x)\sin(x) & -B(x)\sin(x) & +B'(x)\cos(x) & \\ + & y = & A(x)\cos(x) & & +B(x)\sin(x) & & \\ \hline & & & -A'(x)\sin(x) & & +B'(x)\cos(x) & =\csc^2(x) \end{array}$$

$\Rightarrow -A'(x)\sin(x) + B'(x)\cos(x) = \csc^2(x)$

To eliminate one of these terms, we will use this equation in combination with Eq.1

$$\begin{array}{rrcl} & -A'(x)\sin(x) & + & B'(x)\cos(x) & = & \csc^2(x) & \\ + & \tan(x)[A'(x)\cos(x) & + & B'(x)\sin(x)] & = & \tan(x)\cdot 0 & \text{(We multiplied Eq. 1} \\ \hline & & & B'(x)\left(\frac{\sin^2(x)}{\cos(x)} + \cos(x)\right) & = & \csc(x) & \text{by } \tan(x)) \end{array}$$

i.e., $B'(x)\left(\frac{\sin^2(x)}{\cos(x)} + \cos(x)\right) = \csc^2(x)$

$\Rightarrow B'(x)\left(\sin^2(x) + \cos^2(x)\right) = \csc^2(x)\cos(x)$ (We multiplied both sides by $\cos(x)$)

$\Rightarrow B'(x) = \csc(x)\underbrace{\csc(x)\cos(x)}_{=\cot(x)}$

$\Rightarrow B'(x) = \csc(x)\cot(x)$

$\Rightarrow B(x) = \int \csc(x)\cot(x)\,dx = -\csc(x) + c_1$

i.e., $B(x) = -\csc(x) + c_1$

To find $A(x)$, we substitute $B'(x) = \csc(x)\cot(x)$ into Eq. 1 (Our second restriction)

$\Rightarrow A'(x)\cos(x) + \underbrace{\csc(x)\cot(x)}_{B'(x)}\sin(x) = 0$

$\Rightarrow A'(x)\cos(x) + \cot(x) = 0$

$\Rightarrow A'(x)\cos(x) = -\cot(x)$

$\Rightarrow A'(x) = -\frac{1}{\sin(x)}$

$\Rightarrow A'(x) = -\csc(x)$

$\Rightarrow A(x) = -\int \csc(x)\,dx = \ln|\csc(x) - \cot(x)| + c_2$

i.e., $A(x) = \ln|\csc(x) - \cot(x)| + c_2$

Our general solution is:

$y = A(x)\cos(x) + B(x)\sin(x) = (\ln|\csc(x) - \cot(x)| + c_2)\cos(x) + (-\csc(x) + c_1)\sin(x)$

$$\boxed{\begin{aligned} y &= (\ln|\csc(x) - \cot(x)| + c_2)\cos(x) + (-\csc(x) + c_1)\sin(x) \\ &= \ln|\csc(x) - \cot(x)| \cdot \cos(x) - 1 + c_2\cos(x) + c_1\sin(x) \end{aligned}}$$

4. $x^2y'' + 4xy' + 2y = \sin(x)$

First, find the solution to the complementary equation $x^2y'' + 4xy' + 2y = 0$

Our strategy is to seek solutions of the form:

$$y = x^\lambda$$

$$\Rightarrow y' = \lambda x^{\lambda-1}$$

$$\Rightarrow y'' = \lambda(\lambda-1)x^{\lambda-2} = \left(\lambda^2 - \lambda\right)x^{\lambda-2}$$

Plugging these into the complementary equation $x^2y'' + 4xy' + 2y = 0$, we have:

$$x^2\left(\lambda^2 - \lambda\right)x^{\lambda-2} + 4x\lambda x^{\lambda-1} + 2x^\lambda = 0$$

$$\Rightarrow \left(\lambda^2 - \lambda\right)x^\lambda + 4\lambda x^\lambda + 2x^\lambda = 0$$

$$\Rightarrow \left(\lambda^2 - \lambda\right) + 4\lambda + 2 = 0$$

$$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda+1)(\lambda+2) = 0$$

$$\Rightarrow \lambda_1 = -1;\ \lambda_2 = -2$$

Our complementary solution is:

$$y_c = c_1x^{\lambda_1} + c_2x^{\lambda_2} = c_1x^{-1} + c_2x^{-2}$$

Next, we find our particular solution

Since the right hand side of the equation is **not** a linear combination of powers of x, the Method of Undetermined Coefficients will not work. We must use Variation of Parameters.

Thus, we guess that:

$y = A(x)x^{-1} + B(x)x^{-2}$, where the pair $\{A(x), B(x)\}$ is any pair of functions that make $y = A(x)x^{-1} + B(x)x^{-2}$ the general solution to the original differential equation.

This is the first restriction that we impose on the pair $\{A(x), B(x)\}$.

We still have on restriction left to impose.

We will now compute the derivatives of y and plug them into the original equation: $x^2y'' + 4xy' + 2y = \sin(x)$.

$y = A(x)\,x^{-1} + B(x)\,x^{-2}$

$y' = A'(x)\,x^{-1} - A(x)\,x^{-2} + B'(x)\,x^{-2} - 2B(x)\,x^{-3}$

We now impose our second restriction: $A'(x)\,x^{-1} + B'(x)\,x^{-2} = 0$

This results in:

$\Rightarrow y' = -A(x)\,x^{-2} - 2B(x)\,x^{-3}$

$\Rightarrow y'' = -A'(x)\,x^{-2} + 2A(x)\,x^{-3} - 2B'(x)\,x^{-3} + 6B(x)\,x^{-4}$

To find $A(x)$ and $B(x)$ we plug y, y', y'' into the original equation, $x^2y'' + 4xy' + 2y = \sin(x)$.

This yields:

$$\begin{array}{cccccccc} x^2y'' & = & -A'(x) & +2A(x)\,x^{-1} & -2B'(x)\,x^{-1} & +6B(x)\,x^{-2} & \\ +4xy' & = & & -4A(x)\,x^{-1} & & -8B(x)\,x^{-2} & \\ +2y & = & & +2A(x)\,x^{-1} & & +2B(x)\,x^{-2} & \\ \hline x^2y'' + 4xy' + 2y & = & -A'(x) & & -2B'(x)\,x^{-1} & & = \sin(x) \end{array}$$

i.e., $-A'(x) - 2B'(x)\,x^{-1} = \sin(x)$ (Eq. 1)

To eliminate one of the unknown functions, we rely on our second restriction: $A'(x)\,x^{-1} + B'(x)\,x^{-2} = 0$

$\Rightarrow A'(x) + B'(x)\,x^{-1} = 0$

$\Rightarrow A'(x) = -B'(x)\,x^{-1}$

$\Rightarrow -A'(x) = B'(x)\,x^{-1}$ (Eq. 2)

Plugging this into (Eq. 1), we have:

$B'(x)\,x^{-1} - 2B'(x)\,x^{-1} = \sin(x)$

$\Rightarrow -B'(x)\,x^{-1} = \sin(x)$

$\Rightarrow B'(x) = -x\sin(x)$ (Eq. 3)

$$\Rightarrow B(x) = -\int x\sin(x)\,dx = -\int \underbrace{x}_{u}\underbrace{\sin(x)dx}_{dv} = -\left[uv - \int v\,du\right]$$

$$= -\left[x(-\cos(x)) - \int(-\cos(x))\,dx\right] = x\cos(x) - \int \cos(x)\,dx$$

$$= x\cos(x) - \sin(x) + C_2$$

i.e., $B(x) = x\cos(x) - \sin(x) + C_2$

To find $A'(x)$, recall, that from (Eq. 3), $B'(x) = -x\sin(x)$

Plugging this into (Eq. 2), we have:

$-A'(x) = -x\sin(x)\, x^{-1}$

$\Rightarrow A'(x) = x\sin(x)\, x^{-1}$

$\Rightarrow A'(x) = \sin(x)$

$\Rightarrow A(x) = -\cos(x) + C_1$

The solution to the original equation is:

$y = A(x)\, x^{-1} + B(x)\, x^{-2}$

$= (-\cos(x) + C_1)\, x^{-1} + (x\cos(x) - \sin(x) + C_2)\, x^{-2}$

$= -\cos(x)\, x^{-1} + C_1 x^{-1} + x\cos(x)\, x^{-2} - \sin(x)\, x^{-2} + C_2 x^{-2}$

$= -\cos(x)\, x^{-1} + C_1 x^{-1} + \cos(x)\, x^{-1} - \sin(x)\, x^{-2} + C_2 x^{-2}$

$= -\sin(x)\, x^{-2} + C_1 x^{-1} + C_2 x^{-2}$

The solution to the original equation is: $y = -\sin(x)\, x^{-2} + C_1 x^{-1} + C_2 x^{-2}$