Homework #7

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In exercises 1-17, The group $\mathbb{Z}_n$ is the group $(\mathbb{Z}_n, \oplus)$, where $\oplus$ is addition modulo n.

1. List the elements of the group $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
2. Determine whether or not $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is cyclic. If it is cyclic, list the generators.
3. Compute the sum of the elements $(2, 1, 0)$ and $(1, 1, 1)$ in the group $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
4. Compute the sum of the elements $(2, 1, 0)$ and $(2, 1, 1)$ in the group $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
5. List the elements of the group $\mathbb{Z}_6 \times \mathbb{Z}_2$
6. Determine whether or not $\mathbb{Z}_6 \times \mathbb{Z}_2$ is cyclic. If it is cyclic, list the generators.
7. Compute the sum of the elements $(5, 1)$ and $(4, 0)$ in the group $\mathbb{Z}_6 \times \mathbb{Z}_2$
8. Compute the sum of the elements $(3, 1)$ and $(4, 1)$ in the group $\mathbb{Z}_6 \times \mathbb{Z}_2$
9. List the elements of the group $\mathbb{Z}_4 \times \mathbb{Z}_3$
10. Determine whether or not $\mathbb{Z}_4 \times \mathbb{Z}_3$ is cyclic. If it is cyclic, list the generators.
11. Compute the sum of the elements $(3, 1)$ and $(2, 1)$ in the group $\mathbb{Z}_4 \times \mathbb{Z}_3$
12. Compute the sum of the elements $(2, 2)$ and $(2, 2)$ in the group $\mathbb{Z}_4 \times \mathbb{Z}_3$

For exercises, 13-16, use the following facts:

- $\gcd(g, n)$ is the *greatest common divisor* (greatest common factor) of natural numbers g and n.
- $o(g)$ is the order of the element $g \in (\mathbb{Z}_m, +)$ and $o(h)$ is the order of the element $h \in (\mathbb{Z}_n, +)$
- We can compute $o(g)$ by observation or by using the formula in the next bullet point.
- $o(g) = \frac{m}{\gcd(g,m)}$; where m is the order of the group $\mathbb{Z}_m$. (i.e., $m = |\mathbb{Z}_m|$)

- $\text{lcm}(m, n)$ is the *least common multiple* of integers m and n.

- The order of an element $(g, h) \in (\mathbb{Z}_m \times \mathbb{Z}_n, \oplus)$ is given by:

$$o(g, h) = \text{lcm}(o(g), o(h))$$

  where $o(g)$ is the order of the element $g \in (\mathbb{Z}_m, +)$ and $o(h)$ is the order of the element $h \in (\mathbb{Z}_n, +)$.

- Analogously, the order of an element $(g, h, k) \in (\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_p, \oplus)$ is given by:

$$o(g, h, k) = \text{lcm}(o(g), o(h), o(k))$$

  where $o(g)$ is the order of the element $g \in (\mathbb{Z}_m, +)$, $o(h)$ is the order of the element $h \in (\mathbb{Z}_n, +)$, and $o(k)$ is the order of the element $k \in (\mathbb{Z}_p, +)$.

- It may be helpful to know that, for natural numbers m, n, k; $\text{lcm}(m, n, k) = \text{lcm}(\text{lcm}(m, n), k)$

13. Calculate the order of the element $(4, 9)$ in the group $\mathbb{Z}_{18} \times \mathbb{Z}_{18}$

14. Calculate the order of the element $(7, 5)$ in the group $\mathbb{Z}_{12} \times \mathbb{Z}_8$

15. Calculate the order of the element $(8, 6, 4)$ in the group $\mathbb{Z}_{18} \times \mathbb{Z}_9 \times \mathbb{Z}_8$

16. Calculate the order of the element $(8, 6, 4)$ in the group $\mathbb{Z}_9 \times \mathbb{Z}_{17} \times \mathbb{Z}_{10}$

17. Suppose that $(A, *) \leq (G, *)$ and that $(B, *) \leq (H, *)$. Show that $(A \times B, *) \leq (G \times H, *)$.