Exercises Involving Real Numbers #3 - Solutions

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Instructions. Disprove by providing a counter-example.

1. $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

   Consider $a = 9$ and $b = 16$

   Observe: $\sqrt{a+b} = \sqrt{9+16} = 5 \neq 7 = \sqrt{9} + \sqrt{16} = \sqrt{a} + \sqrt{b}$

   i.e., $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ is false, by counter-example.

   Remark 1 *Notice that the selection of perfect squares whose sum is also a perfect square made the result obvious.*

2. $\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2$

   Consider $n = 2$ and $x_1 = 1$ and $x_2 = 1$

   Observe:

   $$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{2} x_i^2 = x_1^2 + x_2^2 = 1^2 + 1^2 = 2$$

   $$\neq 4 = (1+1)^2 = (x_1 + x_2)^2 = \left(\sum_{i=1}^{2} x_i\right)^2 = \left(\sum_{i=1}^{n} x_i\right)^2$$

   i.e., $\sum_{i=1}^{n} x_i^2 = \left(\sum_{i=1}^{n} x_i\right)^2$ is false, by counter-example

3. $x \leq x^2$

   Consider $x = \frac{1}{2}$

   Observe: $x = \frac{1}{2} > \frac{1}{4} = \left(\frac{1}{2}\right)^2 = x^2$

   i.e., $x \leq x^2$ is false, by counter-example

4. $xy \geq x + y$

   Consider: $x = y = \frac{1}{2}$

   Observe: $xy = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4} < \frac{1}{2} + \frac{1}{2} = x + y$

   i.e., $xy \geq x + y$ is false, by counter-example

5. $\frac{1}{x} \leq x$

   Consider $x = \frac{1}{2}$

   Observe: $\frac{1}{x} = \frac{1}{\left(\frac{1}{2}\right)} = 2 > \frac{1}{2} = x$

   i.e., $\frac{1}{x} \leq x$ is false, by counter-example

6. $\frac{1}{x} + \frac{1}{y} \geq \frac{1}{x+y}$

   Consider: $x = 2$ and $y = -1$

   Observe: $\frac{1}{x} + \frac{1}{y} = \frac{1}{(2)} + \frac{1}{(-1)} = \frac{1}{2} - 1 = -\frac{1}{2} < 1 = \frac{1}{2+(-1)} = \frac{1}{x+y}$

   i.e., $\frac{1}{x} + \frac{1}{y} \geq \frac{1}{x+y}$ is false, by counter-example

7. $\frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \frac{x+y}{2}$

   Consider: $x = 1$ and $y = -2$

   Observe: $\frac{2}{\frac{1}{x}+\frac{1}{y}} = \frac{2}{\frac{1}{1}+\frac{1}{(-2)}} = 4 > -\frac{1}{2} = \frac{1+(-2)}{2} = \frac{x+y}{2}$

   i.e., $\frac{2}{\frac{1}{x}+\frac{1}{y}} \leq \frac{x+y}{2}$ is false, by counter-example

8. $|x+y| = |x| + |y|$

   Consider: $x = 1$ and $y = -1$

   Observe: $|x+y| = |1+(-1)| = 0 < 2 = |1| + |-1| = |x| + |y|$

   i.e., $|x+y| = |x| + |y|$ is false, by counter-example

9. $\sqrt{xy} \leq \frac{(x+y)}{2}$

   Consider: $x = -1$ and $y = -1$
   Observe: $\sqrt{xy} = \sqrt{(-1)(-1)} = 1 > -1 = \frac{(-1+(-1))}{2} = \frac{(x+y)}{2}$

   i.e., $\sqrt{xy} \leq \frac{(x+y)}{2}$ is false, by counter-example

10. $xy \leq x|y|$

    Consider: $x = -1$ and $y = -1$

    Observe: $xy = (-1)(-1) = 2 > (-1)|-1| = x|y|$

    i.e., $xy \leq x|y|$ is false, by counter-example

11. $\sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} x_i$

Consider: Let n be any natural number and let $x_i = \frac{1}{2}$ for $i = 1, 2, 3, \ldots, n$

Observe: $\sum_{i=1}^{n} x_i^2 = \underbrace{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^2}_{n \text{ terms}} = \frac{n}{4} < \frac{n}{2} \underbrace{\frac{1}{2} + \frac{1}{2} \ldots + \frac{1}{2}}_{n \text{ terms}} = \sum_{i=1}^{n} x_i$

i.e., $\sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} x_i$ is false, by counter-example

12. $\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 \leq \sum_{i=1}^{n} (x_i + y_i)^2$

Consider:

Let n be any natural number and let $x_i = -1$ and $y_i = 1$ for $i = 1, 2, 3, \ldots, n$

Observe:

$$\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 = \underbrace{(-1)^2 + (-1)^2 + \ldots + (-1)^2}_{n \text{ terms}} + \underbrace{(1)^2 + (1)^2 + \ldots + (1)^2}_{n \text{ terms}} = n + n = 2n$$

$$> 0 = \underbrace{(-1+1)^2 + (-1+1)^2 + \ldots + (-1+1)^2}_{n \text{ terms}} = \sum_{i=1}^{n} (x_i + y_i)^2$$

i.e., $\sqrt{x} \leq x$ is false, by counter-example

13. $\sqrt{x} \leq x$, for all $x \geq 0$.

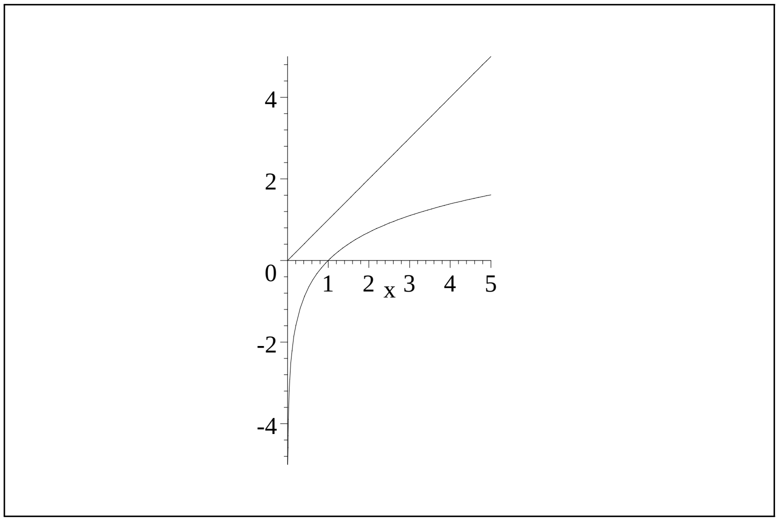
Consider: $x = \frac{1}{4}$

Observe: $\sqrt{x} = \sqrt{\frac{1}{4}} = \frac{1}{2} > \frac{1}{4} = x$

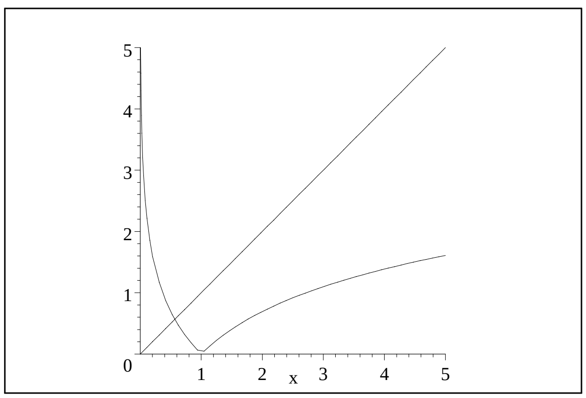
i.e., $\sqrt{x} \leq x$ is false, by counter-example

14. $|\log x| \leq x$, for all $x > 0$.

To get an idea of how to create a counter-example, let's look at the graphs of $y = x$ and $y = \log(x)$ as well as $y = |\log x|$ and $y = x$.



Graphs of $y = x$ and $y = \log(x)$



Graphs of $y = x$ and $y = |\log(x)|$

It appears that for x close to zero, $|\log x| > x$

Let's pick a value of x, close to zero, that makes $\log(x)$ easy to compute.

Consider $x = 0.1 = 10^{-1}$

Observe: $|\log(x)| = |\log(10^{-1})| = |-1| = 1 > 0.1 = x$

i.e., $|\log x| \leq x$ is false, by counter-example.

15. If $x < y$, then $\frac{1}{x} > \frac{1}{y}$ $(x, y \neq 0)$

    Consider: $x = -1$ and $y = 1$

    Observe: $\frac{1}{x} = -1 < 1 = \frac{1}{y}$

    i.e., $\frac{1}{x} > \frac{1}{y}$ is false, by counter-example

16. If $x \leq y$, then $\frac{y}{x} \geq 1$ $(x \neq 0)$

    Consider: $x = -1$ and $y = 1$

    Observe: $\frac{y}{x} = -1 < 1$

    i.e., If $x \leq y$, then $\frac{y}{x} \geq 1$ is false, by counter-example