MTH 4425 - Test #2 - Solutions

Spring 2021

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1. **Prove:** The sequence $\left\{\frac{4}{3}, \frac{7}{5}, \frac{10}{7}, \ldots, \frac{3n+1}{2n+1}, \ldots\right\}$ converges to $L = \frac{3}{2}$.

   Proof. Let $\varepsilon > 0$ be given.

   (We must show that $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $n > N \Rightarrow \underbrace{\left|\frac{3n+1}{2n+1} - \frac{3}{2}\right|}_{|a_n - L|} < \varepsilon$)

   Let $N = N(\varepsilon)$ be given by $\left\lceil \frac{1}{4\varepsilon} \right\rceil$

   Let $n > N$

   Observe:

   $$\underbrace{\left|\frac{3n+1}{2n+1} - \frac{3}{2}\right|}_{|a_n - L|} = \left|\frac{3n+1}{2n+1}\frac{2}{2} - \frac{3}{2}\frac{2n+1}{2n+1}\right| = \left|\frac{-1}{4n+2}\right| = \frac{1}{4n+2} < \frac{1}{4n} < \frac{1}{4N} = \frac{1}{(4)\left\lceil \frac{1}{4\varepsilon} \right\rceil} \leq \frac{1}{4\left(\frac{1}{4\varepsilon}\right)} = \varepsilon$$

   i.e., $n > N \Rightarrow \underbrace{\left|\frac{3n+1}{2n+1} - \frac{3}{2}\right|}_{|a_n - L|} < \varepsilon$

   Hence, The sequence $\left\{\frac{4}{3}, \frac{7}{5}, \frac{10}{7}, \ldots, \frac{3n+1}{2n+1}, \ldots\right\}$ converges to $L = \frac{3}{2}$.

   (i.e., $\lim_{n\to\infty} \frac{3n+1}{2n+1} = \frac{3}{2}$) ■

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   Scratchwork: Given $n > N$, we want $\left|\frac{3n+1}{2n+1} - \frac{3}{2}\right| < \varepsilon$

   $\left|\frac{3n+1}{2n+1} - \frac{3}{2}\right| = \left|\frac{3n+1}{2n+1}\frac{2}{2} - \frac{3}{2}\frac{2n+1}{2n+1}\right| = \left|\frac{-1}{4n+2}\right| = \frac{1}{4n+2} < \frac{1}{4n} < \frac{1}{4N}$

   i.e., $\left|\frac{3n+1}{2n+1} - \frac{3}{2}\right| < \frac{1}{4N}$

   We can make $\left|\frac{3n+1}{2n+1} - \frac{3}{2}\right| < \varepsilon$ by doing the following:

   $\left|\frac{3n+1}{2n+1} - \frac{3}{2}\right| < \frac{1}{4N} \leq \varepsilon$

   This means that $\frac{1}{4\varepsilon} \leq N$

   i.e., $N = \left\lceil \frac{1}{4\varepsilon} \right\rceil$ (The least integer greater than or equal to $\frac{1}{4\varepsilon}$.)

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2. **Prove:** The sequence $\left\{\frac{1}{5}, \frac{2}{8}, \frac{3}{11}, \ldots, \frac{n}{3n+2}, \ldots\right\}$ converges to $L = \frac{1}{3}$.

Proof. Let $\varepsilon > 0$ be given.

(We must show that $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $n > N \Rightarrow \underbrace{\left|\frac{n}{3n+2} - \frac{1}{3}\right|}_{|a_n - L|} < \varepsilon$)

Let $N = N(\varepsilon)$ be given by $\left\lceil \frac{2}{9\varepsilon} \right\rceil$

Let $n > N$

Observe:

$$\underbrace{\left|\frac{n}{3n+2} - \frac{1}{3}\right|}_{|a_n - L|} = \left|\frac{n}{3n+2}\frac{3}{3} - \frac{1}{3}\frac{3n+2}{3n+2}\right| = \left|\frac{3n}{9n+6} - \frac{3n+2}{9n+6}\right| = \left|\frac{-2}{9n+6}\right| = \frac{2}{9n+6} < \frac{2}{9n} < \frac{2}{9N}$$

$$\leq \frac{2}{9\left(\frac{2}{9\varepsilon}\right)} = \varepsilon$$

i.e., $n > N \Rightarrow \underbrace{\left|\frac{n}{3n+2} - \frac{1}{3}\right|}_{|a_n - L|} < \varepsilon$

Hence, The sequence $\left\{\frac{1}{5}, \frac{2}{8}, \frac{3}{11}, \ldots, \frac{n}{3n+2}, \ldots\right\}$ converges to $L = \frac{1}{3}$.

(i.e., $\lim_{n\to\infty} \frac{n}{3n+2} = \frac{1}{3}$) ■

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Scratchwork: Given $n > N$, we want $\left|\frac{n}{3n+2} - \frac{1}{3}\right| < \varepsilon$

$\left|\frac{n}{3n+2} - \frac{1}{3}\right| = \left|\frac{n}{3n+2}\frac{3}{3} - \frac{1}{3}\frac{3n+2}{3n+2}\right| = \left|\frac{3n}{9n+6} - \frac{3n+2}{9n+6}\right| = \left|\frac{-2}{9n+6}\right| = \frac{2}{9n+6} < \frac{2}{9n} < \frac{2}{9N}$

i.e., $\left|\frac{n}{3n+2} - \frac{1}{3}\right| < \frac{2}{9N}$

We can make $\left|\frac{n}{3n+2} - \frac{1}{3}\right| < \varepsilon$ by doing the following:

$\left|\frac{n}{3n+2} - \frac{1}{3}\right| < \frac{2}{9N} \leq \varepsilon$

This means that $\frac{2}{9\varepsilon} \leq N$

i.e., $N = \left\lceil \frac{2}{9\varepsilon} \right\rceil$ (The least integer greater than or equal to $\frac{2}{9\varepsilon}$.)

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3. **Prove:** The sequence $\left\{\frac{1}{6}, \frac{7}{11}, \frac{13}{16}, \ldots, \frac{6n-5}{5n+1}, \ldots\right\}$ converges to $L = \frac{6}{5}$.

Proof. Let $\varepsilon > 0$ be given.

(We must show that $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $n > N \Rightarrow \underbrace{\left|\dfrac{6n-5}{5n+1} - \dfrac{6}{5}\right|}_{|a_n - L|} < \varepsilon$)

Let $N = N(\varepsilon)$ be given by $\left\lceil \frac{31}{25\varepsilon} \right\rceil$

Let $n > N$

Observe:

$$\underbrace{\left|\dfrac{6n-5}{5n+1} - \dfrac{6}{5}\right|}_{|a_n - L|} = \left|\frac{6n-5}{5n+1}\frac{5}{5} - \frac{6}{5}\frac{5n+1}{5n+1}\right| = \left|\frac{30n-25}{25n+5} - \frac{30n+6}{25n+5}\right| = \left|\frac{-31}{25n+5}\right| = \frac{31}{25n+5} < \frac{31}{25n} < \frac{31}{25N}$$

$$\leq \frac{31}{25\left(\frac{31}{25\varepsilon}\right)} = \varepsilon$$

i.e., $n > N \Rightarrow \underbrace{\left|\dfrac{6n-5}{5n+1} - \dfrac{6}{5}\right|}_{|a_n - L|} < \varepsilon$

Hence, The sequence $\left\{\frac{1}{6}, \frac{7}{11}, \frac{13}{16}, \ldots, \frac{6n-5}{5n+1}, \ldots\right\}$ converges to $L = \frac{6}{5}$.

(i.e., $\lim_{n\to\infty} \frac{6n-5}{5n+1} = \frac{6}{5}$) ■

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Scratchwork: Given $n > N$, we want $\left|\frac{6n-5}{5n+1} - \frac{6}{5}\right| < \varepsilon$

$\left|\frac{6n-5}{5n+1} - \frac{6}{5}\right| = \left|\frac{6n-5}{5n+1}\frac{5}{5} - \frac{6}{5}\frac{5n+1}{5n+1}\right| = \left|\frac{30n-25}{25n+5} - \frac{30n+6}{25n+5}\right| = \left|\frac{-31}{25n+5}\right| = \frac{31}{25n+5} < \frac{31}{25n} < \frac{31}{25N}$

i.e., $\left|\frac{6n-5}{5n+1} - \frac{6}{5}\right| < \frac{31}{25N}$

We can make $\left|\frac{6n-5}{5n+1} - \frac{6}{5}\right| < \varepsilon$ by doing the following:

$\left|\frac{6n-5}{5n+1} - \frac{6}{5}\right| < \frac{31}{25N} \leq \varepsilon$

This means that $\frac{31}{25\varepsilon} \leq N$

i.e., $N = \left\lceil \frac{31}{25\varepsilon} \right\rceil$ (The least integer greater than or equal to $\frac{31}{25\varepsilon}$.)

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4. **Prove:** The sequence $\left\{1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots\right\}$ converges to $L = 0$.

Proof. Let $\varepsilon > 0$ be given.

(We must show that $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $n > N \Rightarrow \underbrace{\left|\frac{1}{n^2} - 0\right|}_{|a_n - L|} < \varepsilon$)

Let $N = N(\varepsilon)$ be given by $\left\lceil \frac{1}{\sqrt{\varepsilon}} \right\rceil$

Let $n > N$

Observe:

$$\underbrace{\left|\frac{1}{n^2} - 0\right|}_{|a_n - L|} = \left|\frac{1}{n^2}\right| = \frac{1}{n^2} < \frac{1}{N^2} \leq \frac{1}{\left(\frac{1}{\sqrt{\varepsilon}}\right)^2} = \varepsilon$$

i.e., $n > N \Rightarrow \underbrace{\left|\frac{1}{n^2} - 0\right|}_{|a_n - L|} < \varepsilon$

Hence, The sequence $\left\{1, \frac{1}{4}, \frac{1}{9}, \ldots, \frac{1}{n^2}, \ldots\right\}$ converges to $L = 0$.

(i.e., $\lim_{n \to \infty} \frac{1}{n^2} = 0$) ■

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Scratchwork: Given $n > N$, we want $\left|\frac{1}{n^2} - 0\right| < \varepsilon$

$\left|\frac{1}{n^2} - 0\right| = \left|\frac{1}{n^2}\right| = \frac{1}{n^2} < \frac{1}{N^2}$

i.e., $\left|\frac{1}{n^2} - 0\right| < \frac{1}{N^2}$

We can make $\left|\frac{1}{n^2} - 0\right| < \varepsilon$ by doing the following:

$\left|\frac{1}{n^2} - 0\right| < \frac{1}{N^2} \leq \varepsilon$

This means that $\frac{1}{\varepsilon} \leq N^2$

i.e., $N = \left\lceil \frac{1}{\sqrt{\varepsilon}} \right\rceil$ (The least integer greater than or equal to $\frac{1}{\sqrt{\varepsilon}}$.)

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5. **Prove:** The sequence $\left\{\frac{3}{\sqrt{11}}, \frac{3}{\sqrt{14}}, \frac{3}{\sqrt{19}}, \ldots, \frac{3}{\sqrt{n^2+10}}, \ldots\right\}$ converges to $L = 0$.

Proof. Let $\varepsilon > 0$ be given.

(We must show that $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $n > N \Rightarrow \underbrace{\left|\frac{3}{\sqrt{n^2+10}} - 0\right|}_{|a_n - L|} < \varepsilon$)

Let $N = N(\varepsilon)$ be given by $\left\lceil \frac{3}{\varepsilon} \right\rceil$

Let $n > N$

Observe:

$$\underbrace{\left|\frac{3}{\sqrt{n^2+10}} - 0\right|}_{|a_n - L|} = \left|\frac{3}{\sqrt{n^2+10}}\right| = \frac{3}{\sqrt{n^2+10}} < \frac{3}{\sqrt{n^2}} = \underbrace{\frac{3}{|n|} = \frac{3}{n}}_{\substack{\text{Since } n \to \infty, \\ n \text{ must be positive}}} < \frac{3}{N} \leq \frac{3}{\left(\frac{3}{\varepsilon}\right)} = \varepsilon$$

i.e., $n > N \Rightarrow \underbrace{\left|\frac{3}{\sqrt{n^2+10}} - 0\right|}_{|a_n - L|} < \varepsilon$

Hence, The sequence $\left\{\frac{3}{\sqrt{11}}, \frac{3}{\sqrt{14}}, \frac{3}{\sqrt{19}}, \ldots, \frac{3}{\sqrt{n^2+10}}, \ldots\right\}$ converges to $L = 0$.

(i.e., $\lim_{n\to\infty} \frac{3}{\sqrt{n^2+10}} = 0$) ■

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Scratchwork: Given $n > N$, we want $\left|\frac{3}{\sqrt{n^2+10}} - 0\right| < \varepsilon$

$$\left|\frac{3}{\sqrt{n^2+10}} - 0\right| = \left|\frac{3}{\sqrt{n^2+10}}\right| = \frac{3}{\sqrt{n^2+10}} < \frac{3}{\sqrt{n^2}} = \underbrace{\frac{3}{|n|} = \frac{3}{n}}_{\substack{\text{Since } n \to \infty, \\ n \text{ must be positive}}} < \frac{3}{N}$$

i.e., $\left|\frac{3}{\sqrt{n^2+10}} - 0\right| < \frac{1}{N}$

We can make $\left|\frac{3}{\sqrt{n^2+10}}\right| < \varepsilon$ by doing the following:

$\left|\frac{3}{\sqrt{n^2+10}}\right| < \frac{3}{N} \leq \varepsilon$

This means that $\frac{3}{\varepsilon} \leq N$

i.e., $N = \left\lceil \frac{3}{\varepsilon} \right\rceil$ (The least integer greater than or equal to $\frac{3}{\varepsilon}$.)

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6. Exercises like 1-5

7. The limit of a convergent sequence is unique.

Proof. (By contradiction) Suppose, for the sake of deriving a contradiction, that the sequence $\{a_n\}_{n=1}^{\infty}$ converges and has at least two distinct limits, L_1 and L_2.

Without loss of generality, assume that $L_1 < L_2$. Therefore, there exists an $\varepsilon > 0$ such that $L_2 - L_1 = \varepsilon$.

Since $\{a_n\}_{n=1}^{\infty}$ converges to L_1, $\exists N_1$ such that $n > N_1 \Rightarrow |a_n - L_1| < \frac{\varepsilon}{3}$.

Since $\{a_n\}_{n=1}^{\infty}$ converges to L_2, $\exists N_2$ such that $n > N_2 \Rightarrow |a_n - L_2| < \frac{\varepsilon}{3}$.

Let $N = \max(N_1, N_2)$

Then for $n > N$, we have:

$|a_n - L_1| < \frac{\varepsilon}{3}$ and $|a_n - L_2| < \frac{\varepsilon}{3}$

Consequently, for $n > N$ we have: $\varepsilon = |L_1 - L_2| = |L_1 - a_n + a_n - L_2|$

$\leq |L_1 - a_n| + |a_n - L_2| = |L_1 - a_n| + |L_2 - a_n| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}$

i.e., $\varepsilon < \frac{2\varepsilon}{3}$, a contradiction.

Since the assumption that $\{a_n\}_{n=1}^{\infty}$ has at least two distinct limits, leads to a contradiction, the assumption must be false. Therefore the limit of a convergent sequence is unique. ■

8. **Alternate Proof:** If a sequence converges (i.e., if it has a limit, L, where L is a real number), then the limit is unique.

Proof. Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L_1 and L_2.

Let $\varepsilon > 0$ be given.

Since $\{a_n\}_{n=1}^{\infty}$ converges to L_1, $\exists\ N_1 = N_1(\varepsilon) \in \mathbf{N}$ such that $n > N_1 \Rightarrow |a_n - L_1| < \frac{\varepsilon}{2}$.

Similarly, since $\{a_n\}_{n=1}^{\infty}$ converges to L_2, $\exists\ N_2 = N_2(\varepsilon) \in \mathbf{N}$ such that $n > N_2 \Rightarrow |a_n - L_2| < \frac{\varepsilon}{2}$.

Let $N = \max(N_1, N_2)$.

Then given $n > N$, we have:

$|L_1 - L_2| = |L_1 - a_n + a_n - L_2| = |(L_1 - a_n) + (a_n - L_2)| \leq |L_1 - a_n| + |a_n - L_2| = |L_1 - a_n| + |L_2 - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
i.e., $\forall \varepsilon \in \mathbf{R}$ with $\varepsilon > 0$, $|L_1 - L_2| < \varepsilon$

$\Rightarrow |L_1 - L_2| = 0$

$\Rightarrow L_1 = L_2$ ■

9. (Equivalent criterion for showing that a sequence converges) A sequence $\{a_n\}_{n=1}^{\infty}$ **converges** if to a limit L if and only if $\forall \varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon)$ **contains all but finitely many terms** of the sequence.

Proof.

$\{a_n\}_{n=1}^{\infty}$ converges to $L \Rightarrow$ every open interval of the form $(L - \varepsilon, L + \varepsilon)$ contains all but finitely many terms of the sequence.

Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L, and let $\varepsilon > 0$ be given.

Then $\exists N = N(\varepsilon) \in \mathbf{N}$ such that $n > N \Rightarrow |a_n - L| < \varepsilon$

$\Rightarrow -\varepsilon < a_n - L < \varepsilon$

$\Rightarrow L - \varepsilon < a_n < L + \varepsilon$

i.e., $n > N \Rightarrow a_n \in (L - \varepsilon, L + \varepsilon)$.

Thus the only terms of the sequence that may not be contained in the interval $(L - \varepsilon, L + \varepsilon)$ are $\{a_1, a_2, a_3, \ldots, a_N\}$. Therefore, every open interval of the form $(L - \varepsilon, L + \varepsilon)$ contains all but finitely many terms of the sequence.

Every open interval of the form $(L - \varepsilon, L + \varepsilon)$ contains all but finitely many terms of $\{a_n\}_{n=1}^{\infty} \Rightarrow \{a_n\}_{n=1}^{\infty}$ converges to L.

Let $\varepsilon > 0$ be given and suppose that every open interval of the form $(L - \varepsilon, L + \varepsilon)$ contains all but finitely many terms of the sequence $\{a_n\}_{n=1}^{\infty}$.

Then there are only finitely many terms of the sequence that are *not* contained in the interval $(L - \varepsilon, L + \varepsilon)$.

If all terms of the sequence are contained in the interval $(L - \varepsilon, L + \varepsilon)$, then let $N = 1$.

Otherwise, let N be the largest natural number, such that a_N is not contained in the interval $(L - \varepsilon, L + \varepsilon)$.

Then $n > N \Rightarrow a_n \in (L - \varepsilon, L + \varepsilon)$

$\Rightarrow L - \varepsilon < a_n < L + \varepsilon$

$\Rightarrow -\varepsilon < a_n - L < \varepsilon$

$\Rightarrow |a_n - L| < \varepsilon$

i.e., $\exists N \in \mathbf{N}$ such that $n > N \Rightarrow |a_n - L| < \varepsilon$.

Thus, $\{a_n\}_{n=1}^{\infty}$ converges to L. ■

10. Every convergent sequence of real numbers is bounded.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence with limit L and let $\varepsilon = 1$ be given.

Then, by a previous theorem, the open interval $(L-1, L+1)$ contains all but finitely many terms of the sequence.

If all terms of the sequence are contained in the interval $(L-1, L+1)$, then $L-1 \leq a_n \leq L+1\ \forall n \in \mathbf{N}$, and consequently, the sequence is bounded.

Otherwise, let $\{a_{n_1}, a_{n_2}, \ldots, a_{n_k}\}$ be the terms of the sequence which are not contained in the interval $(L-1, L+1)$. Since there are only finitely many of these terms, there must be a largest and a smallest term. Let M and m be the values of the largest and smallest terms, respectively, of those terms of the sequence that are not contained in the interval $(L-1, L+1)$.

Then, $\min(m, L-1) \leq a_n \leq \max(M, L+1)\ \forall n \in \mathbf{N}$.

Thus, the sequence is bounded. ∎

11. The Monotone Convergence Theorem (a.k.a. The Bounded Convergence Theorem) Every monotone increasing sequence of real numbers, that is bounded above, converges.

Let $\{a_n\}_{n=1}^{\infty}$ be monotone increasing and bounded above. Since the sequence is monotone increasing, it is bounded below (in fact, a_1 is the greatest lower bound).

Since $\{a_n\}_{n=1}^{\infty}$ is bounded, the least upper bound axiom of real numbers guarantees that $\{a_n\}_{n=1}^{\infty}$ has a least upper bound. We'll call it U.

We intend to show that $\{a_n\}_{n=1}^{\infty}$ converges to U.

Let $\varepsilon > 0$ be given.

Since U is the least upper bound for $\{a_n\}_{n=1}^{\infty}$, $\exists N \in \mathbf{N}$ such that $a_N > U - \varepsilon$. (Otherwise, $U - \varepsilon$ would be an upper bound of $\{a_n\}_{n=1}^{\infty}$ that is less than U, contradicting the fact that U is the *least* upper bound.)

Since $\{a_n\}_{n=1}^{\infty}$ is increasing, $n > N \Rightarrow a_n > a_N$.

Thus, $\forall n > N$, we have: $U - \varepsilon < a_N < a_n < U$

i.e., $\forall n > N$, $U - \varepsilon < a_n < U$

$\Rightarrow \forall n > N$, $U - \varepsilon < a_n < U + \varepsilon$

$\Rightarrow \forall n > N$, $-\varepsilon < a_n - U < \varepsilon$

$\Rightarrow \forall n > N$, $|a_n - U| < \varepsilon$

i.e., $n > N \Rightarrow |a_n - U| < \varepsilon$.

Thus, $\{a_n\}_{n=1}^{\infty}$ converges to U. ■

12. (Nested Interval Theorem) Suppose that $\{[a_n, b_n]\}_{n=1}^{\infty} = [a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \ldots$

is a sequence of nested intervals. Then $\exists!$ point p such that $p \in \cap_{n=1}^{\infty} [a_n, b_n]$

Proof. Observe: $\{a_n\}_{n=1}^{\infty}$ is monotone increasing and is bounded above by b_1. By the monotone convergence theorem, $\{a_n\}_{n=1}^{\infty}$ converges to some real number a, where $a = l.u.b.\ \{a_n\}_{n=1}^{\infty}$.

Similarly, $\{b_n\}_{n=1}^{\infty}$ converges to some real number b, where $b = g.l.b.\ \{b_n\}_{n=1}^{\infty}$.

It is our strategy to show that $a = b$. We'll name the common value p. (i.e. $p = a = b$.)

Thus, since $p = l.u.b.\ \{a_n\}_{n=1}^{\infty} = g.l.b.\ \{b_n\}_{n=1}^{\infty}$, it will follow that
$a_n \leq p \leq b_n\ \forall n \in \mathbf{N}$. (i.e., $p \in [a_n, b_n]\ \forall n \in \mathbf{N}$.),
from which it will follow that $p \in \cap_{n=1}^{\infty} [a_n, b_n]$.

Let $\varepsilon > 0$.

In order to show that $a = b$, we need to make some preliminary observations.

Since $\{a_n\}_{n=1}^{\infty}$ converges to a, $\exists N_a \in \mathbf{N}$ such that $n > N_a \Rightarrow |a_n - a| < \frac{\varepsilon}{3}$

Since $\{b_n\}_{n=1}^{\infty}$ converges to b, $\exists N_b \in \mathbf{N}$ such that $n > N_b \Rightarrow |b_n - b| < \frac{\varepsilon}{3}$

Finally, since $\lim_{n\to\infty} (b_n - a_n) = 0$, $\exists N_{ab} \in \mathbf{N}$ such that $n > N_{ab} \Rightarrow |b_n - a_n| < \frac{\varepsilon}{3}$

Let $N = \max(N_a, N_b, N_{ab})$.

Observe: For $n > N$, we have:

$|b - a| = |b - b_n + b_n - a_n + a_n - a| \leq |b - b_n| + |b_n - a_n| + |a_n - a| =$

$|b_n - b| + |b_n - a_n| + |a_n - a| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$

i.e., $\forall \varepsilon > 0$, $|b - a| < \varepsilon$.

Hence, $b - a = 0 \Rightarrow a = b$.

Furthermore, p is unique. For given any $q \in \mathbf{R}$, with $q \neq p$, we have either $q < p$ or $q > p$.

If $q < p$, then $\exists k \in \mathbf{N}$ such that $a_k > q$. (Otherwise, q would be an upper bound of

$\{a_n\}_{n=1}^{\infty}$ that is less than p, contradicting the fact that p is the *least* upper bound of

$\{a_n\}_{n=1}^{\infty}$).

Consequently, $q \notin [a_k, b_k] \Rightarrow q \notin \cap_{n=1}^{\infty} [a_n, b_n]$.

Similarly, if $q > p$, then $q \notin \cap_{n=1}^{\infty} [a_n, b_n]$. ■

13. A point p is a **limit point of a set of real numbers** S, if and only if $\forall\varepsilon > 0$, the interval $(p-\varepsilon, p+\varepsilon)$ contains infinitely many points of S.

Proof. $\boxed{p \text{ is a l.p. of } S \Rightarrow (p-\varepsilon, p+\varepsilon) \text{ contains infinitely many pts of } S}$

Suppose that p is a limit point of S. Then $\forall\varepsilon > 0$, the interval $(p-\varepsilon, p+\varepsilon)$ contains a point of S other than p itself.

Suppose, for the sake of deriving a contradiction, that $\exists\ \varepsilon > 0$ such that $(p-\varepsilon, p+\varepsilon)$ contains only finitely many points $\{s_1, s_2, \ldots, s_n\}$ of S.

For $i = 1, 2, \ldots, n$, let $\delta_i = \underbrace{|s_i - p|}_{\substack{\text{distance from} \\ s_i \text{ to } p}}$

Let $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$

Then the interval $(p-\delta, p+\delta)$ contains no points of S (other than perhaps p itself.)

Hence, p is not a limit point of S, contrary to out hypothesis.

Since the assumption that $\exists$ an open interval $(p-\varepsilon, p+\varepsilon)$ that does not contain infinitely many points of S leads to a contradiction, the assumption must be false.

Hence, every open interval $(p-\varepsilon, p+\varepsilon)$ contains infinitely many points of S.

$\boxed{(p-\varepsilon, p+\varepsilon) \text{ contains infinitely many pts of } S \Rightarrow p \text{ is a l.p. of } S}$

Suppose that $\forall\varepsilon > 0$, the interval $(p-\varepsilon, p+\varepsilon)$ contains infinitely many points of S.

Then every interval $(p-\varepsilon, p+\varepsilon)$ contains at least one point of S other than p itself.

Hence, p is a limit point of S. ■

14. The limit L of a sequence is a limit point of the sequence.

Proof. Suppose that L is the limit of the sequence $\{a_n\}_{n=1}^{\infty}$.

Then $\forall \varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains all but finitely many (hence, infinitely many) terms of the sequence.

Hence, $\forall \varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains infinitely many terms of the sequence.

$\Rightarrow L$ is a limit point of the sequence. ■

15. A convergent sequence can have no limit points other than the limit L.

Proof. (By contradiction)

Suppose, for the sake of deriving a contradiction, that the sequence $\{a_n\}_{n=1}^{\infty}$ has a limit point p that is distinct from the limit L.

Then $\exists\, \varepsilon > 0$ such that $|p - L| = \varepsilon$.

Since L is the limit of the sequence, the interval $\left(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\right)$ contains all but finitely many terms of the sequence.

Since p is a limit point of the sequence, the interval $\left(p - \frac{\varepsilon}{2}, p + \frac{\varepsilon}{2}\right)$ contains infinitely many terms of the sequence.

Observe: No term a_k of the sequence can be contained in both intervals: $\left(p - \frac{\varepsilon}{2}, p + \frac{\varepsilon}{2}\right)$ and $\left(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\right)$.

Otherwise, if a term a_k WERE contained in both intervals, then this would imply that $|a_k - p| < \frac{\varepsilon}{2}$ and $|a_k - L| < \frac{\varepsilon}{2}$.

Thus, we would have:

$$\varepsilon = |p - L| = |(p - a_k) + (a_k - L)| \leq |p - a_k| + |a_k - L| =$$

$$= \underbrace{|a_k - p|}_{<\frac{\varepsilon}{2}} + \underbrace{|a_k - L|}_{<\frac{\varepsilon}{2}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

i.e., $\varepsilon < \varepsilon$, a contradiction.

What does this all mean?

Any term of the sequence that is contained in the interval $\left(p - \frac{\varepsilon}{2}, p + \frac{\varepsilon}{2}\right)$ cannot be contained in the interval $\left(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\right)$.

Since p is a limit point of the sequence, the interval $\left(p - \frac{\varepsilon}{2}, p + \frac{\varepsilon}{2}\right)$ contains infinitely many terms of the sequence.

Since none of these terms can be contained in the interval $\left(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\right)$, this implies that the interval $\left(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\right)$ fails to contain **infinitely** many terms of the sequence.

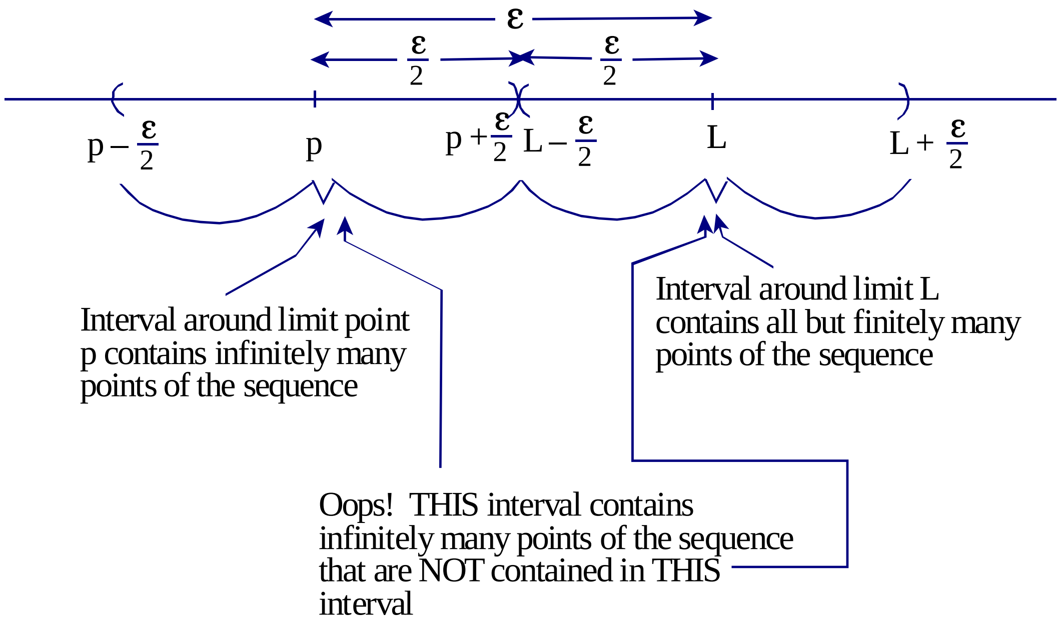
This implies that L is NOT the limit of the sequence, contrary to our hypothesis.

Since the assumption that our convergent sequence has a limit point distinct from the limit yields a contradiction, the assumption must be false.

Hence, a sequence that converges to the limit L can have no other limit point other than L. ■

16. A convergent sequence can have no limit points other than the limit L.

Proof. ("Picture Proof")



17. (Bolzano-Weierstrass) Every bounded set containing infinitely many real numbers has a limit point.

Proof. Let S be a bounded, infinite set of real numbers.

Then S has a g.l.b. and a l.u.b., call them a and b, respectively.

Hence, S is contained in the interval $[a, b]$.

Consider the midpoint $\frac{a+b}{2}$ of the interval. Note that at least one of the intervals, $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$, must contain infinitely many elements of the set S. (Otherwise, both intervals contain only finitely many elements of S, and hence, their union, $[a, b]$ must contain only finitely many elements of set S, contradicting the fact that $[a, b]$ contains S, which is infinite.)

Select an interval (either $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$) which contains infinitely many elements of S, and rename it $[a_1, b_1]$.

Observe that $[a, b] \supseteq [a_1, b_1]$, and that $(b_1 - a_1) = \frac{b-a}{2}$.

Consider the intervals, $\left[a_1, \frac{a_1+b_1}{2}\right]$ and $\left[\frac{a_1+b_1}{2}, b_1\right]$. At least one of these intervals contains infinitely many elements of S. Choose one that does, and rename it $[a_2, b_2]$.

Observe that $[a, b] \supseteq [a_1, b_1] \supseteq [a_2, b_2]$, and that $(b_2 - a_2) = \frac{b_1-a_1}{2} = \frac{b-a}{2^2}$.

Continuing inductively, we generate a sequence of closed, nested intervals:

$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \ldots \supseteq [a_n, b_n] \supseteq \ldots$ where $(b_n - a_n) = \frac{b-a}{2^n}$, and hence, $\lim_{n\to\infty} (b_n - a_n) = 0$.

Thus, $\exists!\ p \in \mathbf{R}$ such that $p \in \cap_{n=1}^{\infty} [a_n, b_n]$.

We claim that p is a limit point of S.

To show this, we must show that $\forall \varepsilon > 0$, the interval $(p - \varepsilon, p + \varepsilon)$ contains infinitely many points of the set S.

So let $\varepsilon > 0$ be given.

Since $\lim_{n\to\infty} (b_n - a_n) = 0$, $\exists m \in \mathbf{N}$ such that $(b_m - a_m) < \varepsilon$

Since $a_m < p < b_m$, we have:

$-p < -a_m$

$\Rightarrow b_m - p < b_m - a_m < \varepsilon$ (we added b_m to both sides)

Hence, $b_m - p < \varepsilon$

$\Rightarrow b_m < p + \varepsilon$

Similarly, since $a_m < p < b_m$, we have:

$p - a_m < b_m - a_m < \varepsilon$ (we subtracted a_m from both sides)

Hence, $p - a_m < \varepsilon$

$\Rightarrow p - \varepsilon < a_m$

Thus, we have: $p - \varepsilon < a_m < b_m < p + \varepsilon$

Hence, interval $[a_m, b_m]$ is contained in the interval $(p - \varepsilon, p + \varepsilon)$.

Since (by hypothesis) $[a_m, b_m]$ contains infinitely many elements of the set S, the interval $(p - \varepsilon, p + \varepsilon)$ contains infinitely many elements of the set S also.

Therefore, p is a limit point of S. ■

18. (Bolzano-Weierstrass Theorem for Sequences) Every bounded sequence (of real numbers) has a limit point.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers.

Then $\{a_n\}_{n=1}^{\infty}$ is a bounded **set** of real numbers and hence, has a g.l.b. and a l.u.b., call them a and b, respectively.

Hence, $\{a_n\}_{n=1}^{\infty}$ is contained in the interval $[a, b]$.

Consider the midpoint $\frac{a+b}{2}$ of the interval. Note that at least one of the intervals, $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$, must contain infinitely many terms of $\{a_n\}_{n=1}^{\infty}$. (Otherwise, both intervals contain only finitely many terms of $\{a_n\}_{n=1}^{\infty}$, and hence, their union, $[a, b]$ must contain only finitely many terms of set $\{a_n\}_{n=1}^{\infty}$, contradicting the fact that $[a, b]$ contains $\{a_n\}_{n=1}^{\infty}$, which has infinitely many terms.)

Select an interval (either $\left[a, \frac{a+b}{2}\right]$ or $\left[\frac{a+b}{2}, b\right]$) which contains infinitely many terms of $\{a_n\}_{n=1}^{\infty}$, and rename it $[a_1, b_1]$.

Observe that $[a, b] \supseteq [a_1, b_1]$, and that $(b_1 - a_1) = \frac{b-a}{2}$.

Consider the intervals, $\left[a_1, \frac{a_1+b_1}{2}\right]$ and $\left[\frac{a_1+b_1}{2}, b_1\right]$. At least one of these intervals contains infinitely many terms of $\{a_n\}_{n=1}^{\infty}$. Choose one that does, and rename it $[a_2, b_2]$.

Observe that $[a, b] \supseteq [a_1, b_1] \supseteq [a_2, b_2]$, and that $(b_2 - a_2) = \frac{b_1-a_1}{2} = \frac{b-a}{2^2}$.

Continuing inductively, we generate a sequence of closed, nested intervals:

$[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq \ldots \supseteq [a_n, b_n] \supseteq \ldots$ where $(b_n - a_n) = \frac{b-a}{2^n}$, and hence, $\lim_{n\to\infty}(b_n - a_n) = 0$.

Thus, $\exists!\ p \in \mathbf{R}$ such that $p \in \cap_{n=1}^{\infty}[a_n, b_n]$.

We claim that p is a limit point of $\{a_n\}_{n=1}^{\infty}$.

To show this, we must show that $\forall \varepsilon > 0$, the interval $(p - \varepsilon, p + \varepsilon)$ contains infinitely many terms of $\{a_n\}_{n=1}^{\infty}$.

So let $\varepsilon > 0$ be given.

Since $\lim_{n\to\infty}(b_n - a_n) = 0$, $\exists m \in \mathbf{N}$ such that $(b_m - a_m) < \varepsilon$

Since $a_m < p < b_m$, we have:

$b_m - p < b_m - a_m < \varepsilon$

Hence, $b_m - p < \varepsilon \Rightarrow b_m < p + \varepsilon$

Similarly, $p - a_m < b_m - a_m < \varepsilon$

Hence, $p - a_m < \varepsilon \Rightarrow p - \varepsilon < a_m$

Thus, we have: $p - \varepsilon < a_m < b_m < p + \varepsilon$

Hence, interval $[a_m, b_m]$ is contained in the interval $(p - \varepsilon, p + \varepsilon)$.

Since (by hypothesis) $[a_m, b_m]$ contains infinitely many terms of $\{a_n\}_{n=1}^{\infty}$, the interval $(p - \varepsilon, p + \varepsilon)$ contains infinitely many terms of $\{a_n\}_{n=1}^{\infty}$ also.

Therefore, p is a limit point of $\{a_n\}_{n=1}^{\infty}$. ■

19. Prove that the set of real numbers in the sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\right\}$ has a limit point.

Proof. Observe: If we let S be the set of distinct real numbers in the sequence, then S is an infinite set of real numbers. Furthermore, $L = 0$ is a lower bound of S and $U = 1$ is an upper bound. Therefore, by the Bolzano-Weierstrass Theorem, the set of real numbers in the sequence has a limit point. ■

20. Prove that the set of real numbers in the sequence $\left\{0, -\frac{1}{2}, \frac{3}{4}, -\frac{7}{8}, \ldots\right\}$ where $a_n = \begin{cases} 1 - \frac{1}{2^{n-1}} & \text{for } n \text{ odd} \\ -1 + \frac{1}{2^{n-1}} & \text{for } n \text{ even} \end{cases}$ has a limit point.

Proof. Observe: If we let S be the set of distinct real numbers in the sequence, then S is an infinite set of real numbers. Furthermore, $L = -1$ is a lower bound of the sequence and $U = 1$ is an upper bound. Therefore, by the Bolzano-Weierstrass Theorem, the set of real numbers in the sequence has a limit point. ■

21. Let S be the set of rational numbers contained in the interval $[0, 1]$. Prove that S has a limit point.

Proof. Observe: Since there are infinitely many rational numbers in any non-trivial interval, the set S contains infinitely many real numbers. Furthermore, $L = 0$ is a lower bound of the sequence and $U = 1$ is an upper bound. Therefore, by the Bolzano-Weierstrass Theorem, S has a limit point. ■

22. Problems like 19 and 21.

23. **Prove:** Every bounded sequence has a convergent subsequence.

(i.e., if $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence, then $\{a_n\}_{n=1}^{\infty}$ has a convergent subsequence.)

Let the hypothesis be given. (i.e., Suppose that $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence.)

Then by the Bolzano-Weierstrass Theorem for Sequences, $\{a_n\}_{n=1}^{\infty}$ has a limit point. Let's call the limit point p.

A consequence of p being a limit point of the sequence is that $\forall \varepsilon > 0$, the interval $(p - \varepsilon, p + \varepsilon)$ contains infinitely many terms of the sequence.

(We will use this fact to construct a sequence that converges to p.)

The interval $(p - 1, p + 1)$, contains infinitely many terms of $\{a_n\}_{n=1}^{\infty}$. Choose one of these terms and call it b_1.

The interval $\left(p - \frac{1}{2}, p + \frac{1}{2}\right)$ contains infinitely many terms of $\{a_n\}_{n=1}^{\infty}$. Since only finitely many of these terms can precede b_1 in the original sequence, the interval $\left(p - \frac{1}{2}, p + \frac{1}{2}\right)$ must contain infinitely many terms of $\{a_n\}_{n=1}^{\infty}$ that succeed b_1 in the original sequence. Choose one such point and call it b_2.

Continuing inductively, the interval $\left(p - \frac{1}{n}, p + \frac{1}{n}\right)$ contains infinitely many terms of $\{a_n\}_{n=1}^{\infty}$. Since only finitely many of these terms can precede b_{n-1} in the original sequence, the interval $\left(p - \frac{1}{n}, p + \frac{1}{n}\right)$ must contain infinitely many terms of $\{a_n\}_{n=1}^{\infty}$ that succeed b_{n-1} in the original sequence. Choose one such point and call it b_n.

Thus, we generate a sequence $\{b_n\}_{n=1}^{\infty}$ having the properties:

i) $b_i \in \{a_n\}_{n=1}^{\infty}, \forall i \in \mathbb{N}$

ii) As terms of the original sequence $\{a_n\}_{n=1}^{\infty}$, b_i precedes b_{i+1}, $\forall i \in \mathbb{N}$

(Thus, $\{b_n\}_{n=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$)

iii) $b_m \in \left(p - \frac{1}{n}, p + \frac{1}{n}\right), \forall m \geq n$

It remains for us to show that $\{b_n\}_{n=1}^{\infty}$ converges to p.

To do this, we will show that $\forall \varepsilon > 0$, the interval $(p - \varepsilon, p + \varepsilon)$ contains all but finitely many terms of the sequence $\{b_n\}_{n=1}^{\infty}$.

So let $\varepsilon > 0$ be given.

Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$ (i.e., $n = \left\lceil \frac{1}{\varepsilon} \right\rceil$).

Then $(p - \varepsilon, p + \varepsilon)$ contains b_m, $\forall m \geq n$.

i.e., $(p - \varepsilon, p + \varepsilon)$ contains all but finitely many terms of $\{b_n\}_{n=1}^{\infty}$.
Thus, $\{b_n\}_{n=1}^{\infty}$ converges to p. ■

24. A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L if and only if every subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to L.

25. Every convergent sequence is a Cauchy Sequence.

26. Every Cauchy Sequence has a limit point.

27. Every Cauchy Sequence is convergent.

28. A sequence $\{a_n\}_{n=1}^{\infty}$ converges if and only it is a Cauchy Sequence.

29. The constant function $f(x) = c$ is continuous

30. Sum, differences, and products of continuous functions are continuous

31. The function $f(x) = x$ is continuous.

32. The function $f(x) = x^n$ is continuous for $n = 0, 1, 2, 3, \ldots$

33. All monomials are continuous

34. All polynomials are continuous

35. Suppose that $f(x)$ and $g(x)$ are defined for all $x \in [a, b]$, except possibly for some point $c \in (a.b)$.

    Suppose also that $f(x) \leq g(x)$ for all $x \in [a, b]$, except possibly for some point $c \in (a.b)$.

    Finally, suppose that $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$, for some $L, M \in \mathbb{R}$.

    Then $L \leq M$. (i.e., $\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x)$)

36. (Sandwich Theorem) Suppose that $f(x), g(x)$, and $h(x)$ are defined for all $x \in [a, b]$, except possibly for some point $c \in (a.b)$.

    Suppose also that $f(x) \leq g(x) \leq h(x)$ for all $x \in [a, b]$, except possibly for some point $c \in (a.b)$.

    Finally, suppose that $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$.

    Then $\lim_{x \to c} g(x) = L$ also.

37. $\lim_{x\to 0} \sin(x) = 0$. ($x$ is measured in radians)

38. $\lim_{x\to 0} \cos(x) = 1$. ($x$ is measured in radians)

39. The function $f(x) = \sin(x)$ is continuous on $(-\infty, \infty)$.

40. The function $f(x) = \cos(x)$ is continuous on $(-\infty, \infty)$.

41. A real-valued function $f(x)$ is continuous at a point $x = L$ if and only if the sequence $\{f(a_n)\}_{n=1}^{\infty}$ converges to $f(L)$, whenever $\{a_n\}_{n=1}^{\infty}$ is a sequence that converges to L.