MTH 4436 HW Set 2.2

Summer 2023

Pat Rossi Name ______________________________

Set 2.2

1. Prove that if a and b are integers, with $b > 0$, then there exist unique integers q and r satisfying:
$$a = qb + r \quad \text{with } 2b \leq r < 3b$$

   Observe: The Division Algorithm guarantees that if a and b are integers, with $b > 0$, then there exist unique integers q' and r' satisfying:
$$a = q'b + r' \quad \text{with } 0 \leq r' < b$$

   If we define $r = r' + 2b$, then $2b \leq r < 3b$.

   The trick now, is to define q such that $a = qb + r \quad$ with $2b \leq r < 3b$.

   To do this, we start with the relationship guaranteed by the Division Algorithm, namely:
$$a = q'b + r' \quad \text{with } 0 \leq r' < b$$

   Since $r = r' + 2b$ (or equivalently, $r' = r - 2b$), we can substitute $r - 2b$ for r'. This yields:
$$a = q'b + (r - 2b) \quad \text{with } 2b \leq r < 3b$$
$$a = (q' - 2)\, b + r \quad \text{with } 2b \leq r < 3b$$

   This suggests that we let $q = q' - 2$. This yields:
$$a = qb + r \quad \text{with } 2b \leq r < 3b$$

2. Show that any integer of the form $6k + 5$ is also of the form $3j + 2$, but not conversely.

   Let $n = 6k + 5$. Then $n = 6k + 5 = 3(2k) + 5 = 3(2k) + 3 + 2 = 3(2k + 1) + 2$.

   Thus, $n = 6k + 5 = 3j + 2$, where $j = 2k + 1$.

   To show that the converse does NOT hold, let $n = 3j + 2$.

   For $j = 2$, we have $n = 3(2) + 2 = 8$

   If $n = 3j + 2 = 6k + 5$, then $n = 3j + 2 = 8 = 6k + 5$.

   But $6k + 5 = 8 \Rightarrow 6k = 3 \Rightarrow k = \frac{1}{2}$, which is not an integer.

   Hence, for $j = 2$, $\; n = 3j + 2 \neq 6k + 5$

3. Use the Division Algorithm to establish the following:

(a) The square of any integer is either of the form $3k$ or $3k+1$.

Let n be an integer. By the Division Algorithm, either

$$n = 3m$$
$$n = 3m + 1$$
$$n = 3m + 2$$

If $n = 3m$, then $n^2 = (3m)^2 = 9m^2 = 3(3m^2) = 3k$, for $k = 3m^2$

If $n = 3m+1$, then $n^2 = (3m+1)^2 = 9m^2+6m+1 = 3(3m^2+2m)+1 = 3k+1$, for $k = 3m^2 + 2m$

If $n = 3m+2$, then $n^2 = (3m+2)^2 = 9m^2+12m+4 = 9m^2+12m+3+1 = 3(3m^2+4m+1)+1 = 3k+1$, for $k = 3m^2+4m+1$.

Hence, for any integer n, n^2 is either of the form $3k$ or $3k+1$.

(b) The cube of any integer has one of the forms, $9k$, $9k+1$, or $9k+8$.

Let n be an integer. By the Division Algorithm, either

$$n = 3m$$
$$n = 3m + 1$$
$$n = 3m + 2$$

If $n = 3m$, then $n^3 = (3m)^3 = 27m^3 = 9(3m^3) = 9k$, for $k = 3m^3$

If $n = 3m+1$, then $n^3 = (3m+1)^3 = 27m^3 + 27m^2 + 9m + 1 = 9(3m^3+3m^2+m)+1 = 9k+1$, for $k = 3m^3+3m^2+m$

If $n = 3m+2$, then $n^3 = (3m+2)^3 = 27m^3+54m^2+36m+8 = 9(3m^3+6m^2+4m)+8 = 9k+8$, for $k = 3m^3+6m^2+4m$

Hence, for any integer n, n^3 has one of the forms, $9k$, $9k+1$, or $9k+8$.

(c) The fourth power of any integer is either of the form $5k$ or $5k+1$.

Let n be an integer. By the Division Algorithm, either

$$\begin{aligned} n &= 5m \\ n &= 5m+1 \\ n &= 5m+2 \\ n &= 5m+3 \\ n &= 5m+4 \end{aligned}$$

If $n = 5m$, then $n^4 = (5m)^4 = 625m^4 = 5\,(125m^4) = 5k$, for $k = 125m^4$

If $n = 5m+1$, then $n^4 = (5m+1)^4 = 625m^4 + 500m^3 + 150m^2 + 20m + 1 = 5\,(125m^4 + 100m^3 + 30m^2 + 4m) + 1 = 5k+1$, for $k = 125m^4 + 100m^3 + 30m^2 + 4m$

If $n = 5m+2$, then $n^4 = (5m+2)^4 = 625m^4 + 1000m^3 + 600m^2 + 160m + 16 =$
$625m^4 + 1000m^3 + 600m^2 + 160m + 15 + 1 =$
$5\,(125m^4 + 200m^3 + 125m^2 + 32m + 3) + 1 =$
$5k+1$, for $k = 125m^4 + 200m^3 + 125m^2 + 32m + 3$

If $n = 5m+3$, then $n^4 = (5m+3)^4 = 625m^4 + 1500m^3 + 1350m^2 + 540m + 81 =$
$625m^4 + 1500m^3 + 1350m^2 + 540m + 80 + 1 =$
$5\,(125m^4 + 300m^3 + 270m^2 + 108m + 16) + 1 =$
$5k+1$, for $k = 125m^4 + 300m^3 + 270m^2 + 108m + 16$

If $n = 5m+4$, then $n^4 = (5m+4)^4 = 625m^4 + 2000m^3 + 2400m^2 + 1280m + 256 =$
$625m^4 + 2000m^3 + 2400m^2 + 1280m + 255 + 1 =$
$5\,(125m^4 + 400m^3 + 480m^2 + 256m + 51) + 1 =$
$5k+1$, for $k = 125m^4 + 400m^3 + 480m^2 + 256m + 51$

Hence, for any integer n, n^4 is either of the form $5k$ or $5k+1$.

4. Prove that $3a^2 - 1$ is never a perfect square.

Observe that $3a^2 - 1 = 3\,(a^2 - 1) + 2 = 3k + 2$, for $k = a^2 - 1$.

The results of problem 3.a tell us that the square of an integer must either be of the form $3k$ or $3k+1$. Hence, $3a^2 - 1 = 3k + 2$ cannot be a perfect square.

5. For $n \geq 1$, prove that $n\left(n+1\right)\left(2n+1\right)/6$ is an integer.

Let n be an integer. By the Division Algorithm, either

$$\begin{aligned} n &= 6m \\ n &= 6m+1 \\ n &= 6m+2 \\ n &= 6m+3 \\ n &= 6m+4 \\ n &= 6m+5 \end{aligned}$$

If $n = 6m$, then $n\left(n+1\right)\left(2n+1\right)/6 = 6m\left(6m+1\right)\left(2\left(6m\right)+1\right)/6 = \left(432m^3 + 108m^2 + 6m\right)/6 = 72m^3 + 18m^2 + m$

i.e., $n\left(n+1\right)\left(2n+1\right)/6$ is an integer, for $n = 6m$

If $n = 6m{+}1$, then $n\left(n+1\right)\left(2n+1\right)/6 = \left(6m+1\right)\left[\left(6m+1\right)+1\right]\left[2\left(6m+1\right)+1\right]/6 = \left(432m^3 + 324m^2 + 78m + 6\right)/6 = 72m^3 + 54m^2 + 13m + 1$

i.e., $n\left(n+1\right)\left(2n+1\right)/6$ is an integer, for $n = 6m+1$

If $n = 6m{+}2$, then $n\left(n+1\right)\left(2n+1\right)/6 = \left(6m+2\right)\left[\left(6m+2\right)+1\right]\left[2\left(6m+2\right)+1\right]/6 = \left(432m^3 + 540m^2 + 222m + 30\right)/6 = 72m^3 + 90m^2 + 37m + 5$

i.e., $n\left(n+1\right)\left(2n+1\right)/6$ is an integer, for $n = 6m+2$

If $n = 6m{+}3$, then $n\left(n+1\right)\left(2n+1\right)/6 = \left(6m+3\right)\left[\left(6m+3\right)+1\right]\left[2\left(6m+3\right)+1\right]/6 = \left(432m^3 + 756m^2 + 438m + 84\right)/6 = 72m^3 + 126m^2 + 73m + 14$

i.e., $n\left(n+1\right)\left(2n+1\right)/6$ is an integer, for $n = 6m+3$

If $n = 6m{+}4$, then $n\left(n+1\right)\left(2n+1\right)/6 = \left(6m+4\right)\left[\left(6m+4\right)+1\right]\left[2\left(6m+4\right)+1\right]/6 = \left(432m^3 + 972m^2 + 726m + 180\right)/6 = 72m^3 + 162m^2 + 121m + 30$

i.e., $n\left(n+1\right)\left(2n+1\right)/6$ is an integer, for $n = 6m+4$

If $n = 6m{+}5$, then $n\left(n+1\right)\left(2n+1\right)/6 = \left(6m+5\right)\left[\left(6m+5\right)+1\right]\left[2\left(6m+5\right)+1\right]/6 = \left(432m^3 + 1188m^2 + 1086m + 330\right)/6 = 72m^3 + 198m^2 + 181m + 55$

i.e., $n\left(n+1\right)\left(2n+1\right)/6$ is an integer, for $n = 6m+5$

Thus, $n\left(n+1\right)\left(2n+1\right)/6$ is an integer for all integers, n.

6. Show that the cube of any integer is of the form $7k$ or $7k \pm 1$.

Let n be an integer. By the Division Algorithm, either

$$\begin{aligned} n &= 7k \\ n &= 7k+1 \\ n &= 7k+2 \\ n &= 7k+3 \\ n &= 7k+4 \\ n &= 7k+5 \\ n &= 7k+6 \end{aligned}$$

If $n = 7m$, then $n^3 = (7m)^3 = 343m^3 = 7\left(49m^3\right) = 7k$.

Hence, if $n = 7m$, then $n^3 = 7k$, for $k = 49m^3$

If $n = 7m+1$, then $n^3 = (7m+1)^3 = 343m^3 + 147m^2 + 21m + 1 =$
$7\left(49m^3 + 21m^2 + 3m\right) + 1 = 7k + 1$.

Hence, if $n = 7m+1$, then $n^3 = 7k+1$, for $k = 49m^3 + 21m^2 + 3m$

If $n = 7m+2$, then $n^3 = (7m+2)^3 = 343m^3 + 294m^2 + 84m + 8 =$
$343m^3 + 294m^2 + 84m + 7 + 1 = 7\left(49m^3 + 42m^2 + 12m + 1\right) + 1 = 7k + 1$.

Hence, if $n = 7m+1$, then $n^3 = 7k+1$, for $k = 49m^3 + 42m^2 + 12m + 1$

If $n = 7m+3$, then $n^3 = (7m+3)^3 = 343m^3 + 441m^2 + 189m + 27 =$
$343m^3 + 441m^2 + 189m + 28 - 1 = 7\left(49m^3 + 63m^2 + 27m + 4\right) - 1 = 7k - 1$.

Hence, if $n = 7m+3$, then $n^3 = 7k-1$, for $k = 49m^3 + 63m^2 + 27m + 4$

If $n = 7m+4$, then $n^3 = (7m+4)^3 = 343m^3 + 588m^2 + 336m + 64 =$
$343m^3 + 588m^2 + 336m + 63 + 1 = 7\left(49m^3 + 84m^2 + 48m + 9\right) + 1 = 7k + 1$.

Hence, if $n = 7m+4$, then $n^3 = 7k+1$, for $k = 49m^3 + 84m^2 + 48m + 9$

If $n = 7m+5$, then $n^3 = (7m+5)^3 = 343m^3 + 735m^2 + 525m + 125 =$
$343m^3 + 735m^2 + 525m + 126 - 1 = 7\left(49m^3 + 105m^2 + 75m + 18\right) - 1 = 7k - 1$.

Hence, if $n = 7m+5$, then $n^3 = 7k-1$, for $k = 49m^3 + 105m^2 + 75m + 18$

If $n = 7m+6$, then $n^3 = (7m+6)^3 = 343m^3 + 882m^2 + 756m + 216 =$
$343m^3 + 882m^2 + 756m + 217 - 1 = 7\left(49m^3 + 126m^2 + 108m + 31\right) - 1 = 7k - 1$.

Hence, if $n = 7m+5$, then $n^3 = 7k-1$, for $k = 49m^3 + 126m^2 + 108m + 31$

Hence, the cube of any integer is of the form $7k$ or $7k \pm 1$.

7. Prove that no integer in the following sequence is a perfect square:

$$11, 111, 1111, 11111, \ldots$$

First, observe that the first term, 11, is not a perfect square.

Next. observe that after the first term of the sequence, a typical term, $111\ldots111$, can be written as

$$111\ldots108 + 3 = 4k + 3$$

By an earlier observation, any perfect square fits either the form $4k$ or the form $4k+1$. Hence, no term in the sequence can be a perfect square.