MTH 1126 - Test #3 - Solutions

Spring 2008

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Name ______________________

Instructions. Show CLEARLY how you arrive at your answers.

1. Compute: $\underbrace{\frac{d}{dx}\left[e^{\cos(x)}\right]}_{\frac{d}{dx}[e^u]} = \underbrace{e^{\cos(x)}}_{e^u} \cdot \underbrace{(-\sin(x))}_{\frac{du}{dx}} = -\sin(x)\, e^{\cos(x)}$

$\boxed{\text{i.e., } \frac{d}{dx}\left[e^{\cos(x)}\right] = -\sin(x)\, e^{\cos(x)}}$

2. Compute: $\int \frac{e^x}{\sqrt{4-e^{2x}}}dx =$

$$\int \frac{e^x}{\sqrt{4-e^{2x}}}dx = \int \underbrace{\frac{1}{\sqrt{(2)^2-(e^x)^2}}}_{\frac{1}{\sqrt{a^2-u^2}}} \underbrace{e^x dx}_{du} = \int \frac{1}{\sqrt{a^2-u^2}}du = \sin^{-1}\left(\frac{u}{a}\right) = \sin^{-1}\left(\frac{e^x}{2}\right) + C$$

$$\boxed{\begin{aligned} a^2 &= 4 \\ a &= 2 \\ u^2 &= e^{2x} = (e^x)^2 \\ u &= e^x \\ \frac{du}{dx} &= e^x \\ du &= e^x dx \end{aligned}}$$

$\boxed{\int \frac{e^x}{\sqrt{4-e^{2x}}}dx = \sin^{-1}\left(\frac{e^x}{2}\right) + C}$

3. Given that $\ln(2) \approx 0.7$ and $\ln(5) \approx 1.6$, approximate the following:

(a) $\ln(10) = \ln(2\cdot 5) = \ln(2) + \ln(5) \approx 0.7 + 1.6 = 2.3$

$\boxed{\ln(10) \approx 2.3}$

(b) $\ln(50) = \ln(2\cdot 5^2) = \ln(2) + \ln(5^2) = \ln(2) + 2\ln(5) \approx 0.7 + 2(1.6) = 3.9$

$\boxed{\ln(50) \approx 3.9}$

4. $\int e^{3x^2} x\,dx =$

$$\int \underbrace{e^{3x^2}}_{e^u} \underbrace{x\,dx}_{\frac{1}{6}du} = \int e^u \frac{1}{6} du = \frac{1}{6}\int e^u du = \frac{1}{6}e^u + C = \frac{1}{6}e^{3x^2} + C$$

$$\boxed{\begin{array}{lcl} u & = & 3x^2 \\ \frac{du}{dx} & = & 6x \\ du & = & 6x\,dx \\ \frac{1}{6}du & = & x\,dx \end{array}}$$

$$\boxed{\text{i.e., } \int e^{3x^2} x\,dx = \frac{1}{6}e^{3x^2} + C}$$

5. Compute: $\int \frac{\ln(\sqrt{x})}{\sqrt{x}}\,dx = \int \frac{\ln\left(x^{\frac{1}{2}}\right)}{x^{\frac{1}{2}}}\,dx = \int \underbrace{\ln\left(x^{\frac{1}{2}}\right)}_{\ln(u)} \underbrace{x^{-\frac{1}{2}}dx}_{2du} = \int \ln(u)\, 2du = \underbrace{2\int \ln(u)\,du}_{\text{If you got this far, you got full credit.}}$

$$= 2\left[u\ln(u) - u\right] + C = 2\left(x^{\frac{1}{2}}\ln\left(x^{\frac{1}{2}}\right) - x^{\frac{1}{2}}\right) + C$$

$$\begin{array}{lcl} u & = & x^{\frac{1}{2}} \\ \frac{du}{dx} & = & \frac{1}{2}x^{-\frac{1}{2}} \\ du & = & \frac{1}{2}x^{-\frac{1}{2}}dx \\ 2du & = & x^{-\frac{1}{2}}dx \end{array}$$

$$\boxed{\int \frac{\ln(\sqrt{x})}{\sqrt{x}}\,dx = 2\left(x^{\frac{1}{2}}\ln\left(x^{\frac{1}{2}}\right) - x^{\frac{1}{2}}\right) + C}$$

Alternatively: Use Integration by Parts:

$$\int \frac{\ln(\sqrt{x})}{\sqrt{x}}\,dx = \int \frac{\ln\left(x^{\frac{1}{2}}\right)}{x^{\frac{1}{2}}}\,dx = \int \underbrace{\ln\left(x^{\frac{1}{2}}\right)}_{u} \underbrace{x^{-\frac{1}{2}}dx}_{dv} = uv - \int v\,du$$

$$= \underbrace{\ln\left(x^{\frac{1}{2}}\right)}_{u} \cdot \underbrace{2x^{\frac{1}{2}}}_{v} - \int \underbrace{2x^{\frac{1}{2}}}_{v} \cdot \underbrace{\frac{1}{2x}dx}_{du} = 2x^{\frac{1}{2}}\ln\left(x^{\frac{1}{2}}\right) - \int x^{-\frac{1}{2}}dx = 2x^{\frac{1}{2}}\ln\left(x^{\frac{1}{2}}\right) - 2x^{\frac{1}{2}} + C$$

$$\boxed{\begin{array}{lcl@{\qquad\qquad}lcl} u & = & \ln\left(x^{\frac{1}{2}}\right) & dv & = & x^{-\frac{1}{2}}dx \\ \frac{du}{dx} & = & \frac{1}{x^{\frac{1}{2}}} \cdot \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x} & \int dv & = & \int x^{-\frac{1}{2}}dx \\ du & = & \frac{1}{2x}dx & v & = & 2x^{\frac{1}{2}} \end{array}}$$

$$\boxed{\int \frac{\ln(\sqrt{x})}{\sqrt{x}}\,dx = 2\left(x^{\frac{1}{2}}\ln\left(x^{\frac{1}{2}}\right) - x^{\frac{1}{2}}\right) + C}$$

6. Compute: $\int x \ln(x)\ dx =$

Use Integration by Parts:

$$\int x \ln(x)\ dx = \int \underbrace{\ln(x)}_{u} \underbrace{x\,dx}_{dv} = \int udv = uv - \int vdu = \underbrace{\ln(x)}_{u} \cdot \underbrace{\frac{1}{2}x^2}_{v} - \int \underbrace{\frac{1}{2}x^2}_{v} \cdot \underbrace{\frac{1}{x}dx}_{du}$$

$$= \tfrac{1}{2}x^2 \ln(x) - \tfrac{1}{2}\int xdx = \tfrac{1}{2}x^2 \ln(x) - \tfrac{1}{2}\tfrac{x^2}{2} + C = \tfrac{1}{2}x^2 \ln(x) - \tfrac{1}{4}x^2 + C$$

u	$=$	$\ln(x)$	dv	$=$	xdx
$\frac{du}{dx}$	$=$	$\frac{1}{x}$	$\int dv$	$=$	$\int xdx$
du	$=$	$\frac{1}{x}dx$	v	$=$	$\frac{1}{2}x^2$

i.e. $\int x \ln(x)\ dx = \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C$

7. Compute: $\underbrace{\dfrac{d}{dx}[\ln(\sin(x))]}_{\frac{d}{dx}[\ln(u)]} = \underbrace{\dfrac{1}{\sin(x)}}_{\frac{i}{u}} \cdot \underbrace{\cos(x)}_{\frac{du}{dx}} = \frac{\cos(x)}{\sin(x)} = \cot(x)$

i.e., $\frac{d}{dx}[\ln(\sin(x))] = \cot(x)$

8. $\underbrace{\dfrac{d}{dx}\left[\tan^{-1}(\sin(x))\right]}_{\frac{d}{dx}[\tan^{-1}(u)]} = \underbrace{\dfrac{1}{1+\left(\sin^2(x)\right)^2}}_{\frac{1}{1+u^2}} \cdot \underbrace{\cos(x)}_{\frac{du}{dx}} = \frac{\cos(x)}{1+\sin^2(x)}$

i.e., $\frac{d}{dx}[\tan^{-1}(\sin(x))] = \frac{\cos(x)}{1+\sin^2(x)}$

9. $\int \sin^3(x) \cos^4(x)\ dx =$

i. Pull out a factor of $\sin(x)$ to serve as the “future du.”

$= \int \sin^2(x) \cos^4(x) \underbrace{\sin(x)\ dx}_{\text{“future dú}}$

Note: We intend to let $u = \cos(x)$. Consequently, $du = -\sin(x)$

Convert remaining sines into cosines by

ii. Re-writing sines in terms of $\sin^2(x)$

Done.

iii. Replace $\sin^2(x)$ with $1-\cos^2(x)$

$$= \int \underbrace{\left(1-\cos^2(x)\right)\cos^4(x)}_{(1-u^2)u^4}\underbrace{\sin(x)\,dx}_{-du} = \int (1-u^2)\,u^4\,(-du) = \int (u^6-u^4)\,du$$

$$= \tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + C = \tfrac{1}{7}\cos^7(x) - \tfrac{1}{5}\cos^5(x) + C$$

$$\boxed{\text{i.e., } \int \sin^3(x)\cos^4(x)\,dx = \tfrac{1}{7}\cos^7(x) - \tfrac{1}{5}\cos^5(x) + C}$$