MTH 3318 Solutions to Induction Problems - Set 1a

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Instructions. Prove the following by mathematical induction.

1. $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$; $\forall n \in \mathbf{N}$

   i.e. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

   Proof.

   Step #1: Show that the proposition is true for $n = 1$.

   $\sum_{i=1}^{1} i = 1 = \frac{(1)((1)+1)}{2}$ True.

   Step #2: Assume that the proposition is true for $n = k$, and prove that the proposition is true for $n = k + 1$.

   i.e., Assume that $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ is true for some natural number k, and prove that $\sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$ is true. (Equivalently, prove that $\sum_{i=1}^{k+1} i = \frac{k^2+3k+2}{2}$.)

   Observe:

   $$\sum_{i=1}^{k+1} i = \underbrace{\sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1)}_{\text{by Induction Hypothesis}} = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

   $$= \frac{k^2+k}{2} + \frac{2k+2}{2} = \frac{k^2+3k+2}{2}.$$

   i.e., $\sum_{i=1}^{k+1} i = \frac{k^2+3k+2}{2}$.

   Hence, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for all natural numbers, n. ■

2. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$; $\forall n \in \mathbf{N}$

i.e. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Proof.

Step #1: Show that the proposition is true for $n = 1$.

$\sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{(1)[(1)+1][2(1)+1]}{6}$. True.

Step #2: Assume that the proposition is true for $n = k$, and prove that the proposition is true for $n = k + 1$.

i.e., Assume that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ and show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$.

i.e., show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Observe:

$$\sum_{i=1}^{k+1} i^2 = \underbrace{\sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2}_{\text{by Induction Hypothesis}} = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)\left[2k^2+7k+6\right]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

i.e., $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Hence, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$; $\forall n \in \mathbf{N}$ ■

3. $1 + x + x^2 + x^3 + \ldots + x^n = \frac{x^{n+1}-1}{x-1}$; $\forall n \in \mathbf{N} \cup \{0\}$; where $x \neq 1$.

i.e. $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$; where $x \neq 1$.

Proof.

Step #1: Show that the proposition is true for $n = 0$.

$\sum_{i=0}^{0} x^i = x^{(0)} = 1 = \frac{x-1}{x-1} = \frac{x^{(0)+1}-1}{x-1}$. True.

Step #2: Assume that the proposition is true for $n = k$, and prove that the proposition is true for $n = k + 1$.

i.e., Assume that $\sum_{i=0}^{k} x^i = \frac{x^{k+1}-1}{x-1}$ and show that $\sum_{i=0}^{k+1} x^i = \frac{x^{(k+1)+1}-1}{x-1}$.

i.e., show that $\sum_{i=0}^{k+1} x^i = \frac{x^{k+2}-1}{x-1}$

Observe:

$$\sum_{i=0}^{k+1} x^i = \underbrace{\sum_{i=0}^{k} x^i + x^{k+1} = \frac{x^{k+1}-1}{x-1} + x^{k+1}}_{\text{by Induction Hypothesis}} = \frac{x^{k+1}-1}{x-1} + \frac{x^{k+1}(x-1)}{x-1}$$

$$= \frac{x^{k+1}-1}{x-1} + \frac{x^{k+2}-x^{k+1}}{x-1} = \frac{x^{k+1}-1+x^{k+2}-x^{k+1}}{x-1} = \frac{x^{k+2}-1}{x-1}$$

i.e., $\sum_{i=0}^{k+1} x^i = \frac{x^{k+2}-1}{x-1}$

Hence, $\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1}$; $\forall n \in \mathbf{N} \cup \{0\}$ ■

Remark: In this example, we allowed 0 to be an " honorary member" of the natural numbers. If we want to prove a proposition $P(n)$ $\forall n \in \mathbf{N} \cup \{0\}$, we prove $P(n)$ true for $n = 0$ on the first induction step.

4. $1 + 3 + 5 + \ldots + (2n-1) = n^2$

i.e. $\sum_{i=1}^{n} (2i-1) = n^2$

Proof.

Step #1: Show true for $n = 1$

$\sum_{i=1}^{1} (2i-1) = (2(1)-1) = 1 = (1)^2$ True.

Step #2: Assume true for $n = k$, and show true for $n = k+1$

i.e., Assume that $\sum_{i=1}^{k} (2i-1) = k^2$ for some natural number k, and show that $\sum_{i=1}^{k+1} (2i-1) = (k+1)^2$

Observe:

$$\sum_{i=1}^{k+1} (2i-1) = \underbrace{\sum_{i=1}^{k} (2i-1) + (2(k+1)-1) = k^2 + (2(k+1)-1)}_{\text{by Induction Hypothesis}}$$

$$= k^2 + 2k + 1 = (k+1)^2$$

i.e., $\sum_{i=1}^{k+1} (2i-1) = (k+1)^2$

Hence, $\sum_{i=1}^{n} (2i-1) = n^2$ for all natural numbers, n. ■

5. $2 + 4 + 6 + \ldots + 2n = n^2 + n; \forall n \in \mathbf{N}$

i.e. $\sum_{i=1}^{n} 2i = n^2 + n$

Proof.

Step #1: Show true for $n = 1$

$\sum_{i=1}^{1} 2i = 2(1) = 2 = (1)^2 + (1)$ True.

Step #2: Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $\sum_{i=1}^{k} 2i = k^2 + k$ for some natural number k, and show that $\sum_{i=1}^{k+1} 2i = (k+1)^2 + (k+1)$

Observe:

$$\sum_{i=1}^{k+1} 2i = \underbrace{\sum_{i=1}^{k} 2i + 2(k+1) = \left(k^2 + k\right) + 2(k+1)}_{\text{by Induction Hypothesis}}$$

$$= (k^2 + k) + 2k + 2 = k^2 + k + (k+1) + (k+1) = k^2 + 2k + 1 + (k+1)$$

$$= (k+1)^2 + (k+1)$$

i.e., $\sum_{i=1}^{k+1} 2i = (k+1)^2 + (k+1)$

Hence, $\sum_{i=1}^{n} 2i = n^2 + n$ for all natural numbers, n. ∎

6. $1 + 5 + 9 + \ldots + (4n - 3) = 2n^2 - n$

i.e., $\sum_{i=1}^{n} (4i - 3) = 2n^2 - n$

Proof.

Step #1: Show true for $n = 1$

$\sum_{i=1}^{1} (4i - 3) = 4(1) - 3 = 1 = 2(1)^2 - (1)$ True.

Step #2: Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $\sum_{i=1}^{k} (4i - 3) = 2k^2 - k$ for some natural number k, and show that $\sum_{i=1}^{k+1} (4i - 3) = 2(k+1)^2 - (k+1)$

Equivalently, show that $\sum_{i=1}^{k+1} (4i - 3) = 2k^2 + 3k + 1$

Observe:

$$\sum_{i=1}^{k+1} (4i - 3) = \underbrace{\sum_{i=1}^{k} (4i - 3) + 4[(k+1) - 3] = \left(2k^2 - k\right) + 4[(k+1) - 3]}_{\text{by Induction Hypothesis}}$$

$$= (2k^2 - k) + 4k + 4 - 3$$

$$= 2k^2 + 3k + 1$$

i.e., $\sum_{i=1}^{k+1} (4i - 3) = 2k^2 + 3k + 1$

Hence, $\sum_{i=1}^{n} (4i - 3) = 2n^2 - n$ for all natural numbers, n. ■

7. $3 + 7 + 11 + \ldots + (4n - 1) = 2n^2 + n$

i.e., $\sum_{i=1}^{n} (4i - 1) = 2n^2 + n$

Proof.

Step #1: Show true for $n = 1$

$\sum_{i=1}^{1} (4i - 1) = (4(1) - 1) = 3 = 2(1)^2 + (1)$ True.

Step #2: Assume true for $n = k$, and show true for $n = k + 1$

i.e., Assume that $\sum_{i=1}^{k} (4i - 1) = 2k^2 + k$ for some natural number k, and show that $\sum_{i=1}^{k+1} (4i - 1) = 2(k + 1)^2 + (k + 1)$

Equivalently, show that $\sum_{i=1}^{k+1} (4i - 1) = 2k^2 + 5k + 3$

Observe:

$$\sum_{i=1}^{k+1} (4i - 1) = \underbrace{\sum_{i=1}^{k} (4i - 1) + [4(k + 1) - 1] = \left(2k^2 + k\right) + [4(k + 1) - 1]}_{\text{by Induction Hypothesis}}$$

$$= 2k^2 + k + 4k + 4 - 1$$

$$= 2k^2 + 5k + 3$$

i.e., $\sum_{i=1}^{k+1} (4i - 1) = 2k^2 + 5k + 3$

Hence, $\sum_{i=1}^{n} (4i - 1) = 2n^2 + n$ for all natural numbers, n. ■

8. $2+6+10+\ldots+(4n-2)=2n^2$

i.e., $\sum_{i=1}^{n}(4i-2)=2n^2$

Proof.

Step #1: Show true for $n=1$

$\sum_{i=1}^{1}(4i-2)=(4(1)-2)=2=2(1)^2$ True.

Step #2: Assume true for $n=k,$ and show true for $n=k+1$

i.e., Assume that $\sum_{i=1}^{k}(4i-2)=2k^2$ for some natural number k, and show that $\sum_{i=1}^{k+1}(4i-2)=2(k+1)^2$

Observe:

$$\sum_{i=1}^{k+1}(4i-2)=\underbrace{\sum_{i=1}^{k}(4i-2)+[4(k+1)-2]=2k^2+[4(k+1)-2]}_{\text{by Induction Hypothesis}}$$

$$=2k^2+4k+4-2$$

$$=2k^2+4k+2$$

$$=2(k^2+2k+1)$$

$$=2(k+1)^2$$

i.e., $\sum_{i=1}^{k+1}(4i-2)=2(k+1)^2$

Hence, $\sum_{i=1}^{n}(4i-2)=2n^2$ for all natural numbers, n. ■

9. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

i.e. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Proof.

Step #1: Show true for $n = 1$

$\sum_{i=1}^{1} i^3 = (1)^3 = 1 = \frac{(1)^2((1)+1)^2}{4}$ True.

Step #2: Assume true for $n = k,$ and show true for $n = k + 1$

i.e., Assume that $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$ for some natural number k, and show that

$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$

i.e., $\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$

Observe:

$$\sum_{i=1}^{k+1} i^3 = \underbrace{\sum_{i=1}^{k} i^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3}_{\text{by Induction Hypothesis}} = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$$= \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right] = \frac{(k+1)^2}{4}\left[k^2 + 4k + 4\right] = \frac{(k+1)^2(k+2)^2}{4}$$

i.e., $\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$

Hence, $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ for all natural numbers, n. ■