Integrals and Natural Logarithms #4 - Solutions

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Instructions

Answers appear on the ANSWERS page. Solutions appear on the SOLUTIONS page.

1. Compute: $\int \left(15x^4 + 12x^3 + 8x + 3\sqrt{x} + 2\right) dx =$

(Re-write) $\int \left(15x^4 + 12x^3 + 8x + 3x^{\frac{1}{2}} + 2\right) dx = 15\left[\frac{x^5}{5}\right] + 12\left[\frac{x^4}{4}\right] + 8\left[\frac{x^2}{2}\right] + 3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right] + 2x + C$

$= 3x^5 + 3x^4 + 4x^2 + 2x^{\frac{3}{2}} + 2x + C$

> i.e., $\int \left(15x^4 + 12x^3 + 8x + 3\sqrt{x} + 2\right) dx = 3x^5 + 3x^4 + 4x^2 + 2x^{\frac{3}{2}} + 2x + C$
> Don't forget the "+C"

2. Compute: $\int (4\sin(x) + 3\csc(x)\cot(x))\, dx =$

$\int (4\sin(x) + 3\csc(x)\cot(x))\, dx = 4[-\cos(x)] + 3[-\csc(x)] + C$

> i.e., $\int (4\sin(x) + 3\csc(x)\cot(x))\, dx = -4\cos(x) - 3\csc(x) + C$
> Don't forget the "+C"

3. Compute: $\int_{x=0}^{x=2} \left(6x^2 + 9x + 1\right) dx =$

$$\int_{x=0}^{x=2} \underbrace{\left(6x^2 + 9x + 1\right)}_{f(x)} dx = \underbrace{\left[2x^3 + \frac{9}{2}x^2 + x\right]_{x=0}^{x=2}}_{F(x)}$$

$$= \underbrace{\left[2(2)^3 + \frac{9}{2}(2)^2 + (2)\right]}_{F(2)} - \underbrace{\left[2(0)^3 + \frac{9}{2}(0)^2 + (0)\right]}_{F(0)} = 36$$

> i.e., $\int_{x=0}^{x=2} \left(6x^2 + 9x + 1\right) dx = 36$

4. Compute: $\int \left(5x^2+10x+2\right)^{10}(x+1)\,dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(5x^2+10x+2\right)^{10}$ (A function raised to a power is always a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(5x^2+10x+2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(5x^2+10x+2\right)}_{\text{function}} - - - - \rightarrow \underbrace{(x+1)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(5x^2+10x+2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 5x^2+10x+2 \\ \Rightarrow \frac{du}{dx} &= 10x+10 \\ \Rightarrow du &= (10x+10)\,dx \\ \Rightarrow \frac{1}{10}du &= (x+1)\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(5x^2+10x+2\right)^{10}}_{u^{10}}\underbrace{(x+1)\,dx}_{\frac{1}{10}du} = \int u^{10}\frac{1}{10}du = \frac{1}{10}\int u^{10}du$$

4. Integrate (in terms of u).

$$\frac{1}{10}\int u^{10}du = \frac{1}{10}\left[\frac{u^{11}}{11}\right] + C = \frac{1}{110}u^{11} + C$$

5. Re-express in terms of the original variable, x.

$$\int \left(5x^2+10x+2\right)^{10}(x+1)\,dx = \underbrace{\frac{1}{110}\left(5x^2+10x+2\right)^{11} + C}_{\frac{1}{110}u^{11}+C}$$

i.e., $\int \left(5x^2+10x+2\right)^{10}(x+1)\,dx = \frac{1}{110}\left(5x^2+10x+2\right)^{11} + C$

5. Compute: $\int \cos\left(8x^3+3x^2\right)\left(4x^2+x\right)dx =$

   1. Is u-sub appropriate?

      a. Is there a composite function?

      Yes! $\cos\left(8x^3+3x^2\right)$

      outer ↗ ↖ inner

      Let $u =$ the "inner" of the composite function

      $\Rightarrow u = 8x^3+3x^2$

      b. Is there an (approximate) function/derivative pair?

      Yes! $\underbrace{\left(8x^3+3x^2\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(4x^2+x\right)}_{\text{deriv}}$

      Let $u =$ the "function" of the function/deriv pair

      $\Rightarrow u = 8x^3+3x^2$

      c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

      (i.e., do criteria **a** and **b** suggest the same choice of u ?)

      Yes! $\Rightarrow$ u-substitution is appropriate

   2. Compute du

$$\begin{aligned} u &= 8x^3+3x^2 \\ \Rightarrow \frac{du}{dx} &= 24x^2+6x \\ \Rightarrow du &= \left(24x^2+6x\right)dx \\ \Rightarrow \frac{1}{6}du &= \left(4x^2+x\right)dx \end{aligned}$$

   3. Analyze in terms of u and du

$$\int \underbrace{\cos\left(8x^3+3x^2\right)}_{\cos(u)}\underbrace{\left(4x^2+x\right)dx}_{\frac{1}{6}du} = \int \cos(u)\ \frac{1}{6}du = \frac{1}{6}\int \cos(u)\,du$$

   4. Integrate (in terms of u).

$$\frac{1}{6}\int \cos(u)\,du = \frac{1}{6}\left[\sin(u)\right]+C = \frac{1}{6}\sin(u)+C$$

   5. Re-express in terms of the original variable, x.

$$\int \cos\left(8x^3+3x^2\right)\left(4x^2+x\right)dx = \underbrace{\frac{1}{6}\sin\left(8x^3+3x^2\right)+C}_{\frac{1}{6}\sin(u)+C}$$

i.e., $\int \cos\left(8x^3+3x^2\right)\left(4x^2+x\right)dx = \frac{1}{6}\sin\left(8x^3+3x^2\right)+C$

6. Compute: $\int \frac{\cos(x)}{\sin(x)+5} dx =$

$$\int \frac{\cos(x)}{\sin(x)+5} dx \underbrace{=}_{\text{re-write}} \int \frac{1}{\sin(x)+5}\ \cos(x)\ dx$$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{\sin(x)+5}$ is the same as $(\sin(x)+5)^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = (\sin(x)+5)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{(\sin(x)+5)}_{\text{function}} - - - - \rightarrow \underbrace{\cos(x)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = (\sin(x)+5)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= \sin(x)+5 \\ \Rightarrow \frac{du}{dx} &= \cos(x) \\ \Rightarrow du &= \cos(x)\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\frac{1}{\sin(x)+5}}_{\frac{1}{u}} \underbrace{\cos(x)\ dx}_{du} = \int \frac{1}{u} du$$

4. Integrate (in terms of u).

$$\int \frac{1}{u} du = \ln|u| + C$$

5. Re-express in terms of the original variable, x.

$$\int \frac{\cos(x)}{\sin(x)+5} dx = \underbrace{\ln|\sin(x)+5| + C}_{\ln|u|+C}$$

i.e., $\int \frac{\cos(x)}{\sin(x)+5} dx = \ln|\sin(x)+5| + C$

7. Compute: $\frac{d}{dx}\left[\ln\left(\sec\left(x\right)\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(\sec\left(x\right)\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{\sec\left(x\right)}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(\sec\left(x\right)\tan\left(x\right)\right)}_{g'(x)} = \frac{\sec(x)\tan(x)}{\sec(x)} = \tan\left(x\right)$$

i.e., $\frac{d}{dx}\left[\ln\left(\sec\left(x\right)\right)\right] = \frac{\sec(x)\tan(x)}{\sec(x)} = \tan\left(x\right)$

8. Compute: $\frac{d}{dx}\left[\ln\left(10x^3 - 8x^2 + 4x\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(10x^3 - 8x^2 + 4x\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{10x^3 - 8x^2 + 4x}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(30x^2 - 16x + 4\right)}_{g'(x)} = \frac{30x^2-16x+4}{10x^3-8x^2+4x} = \frac{15x^2-8x+2}{5x^3-4x^2+2x}$$

i.e., $\frac{d}{dx}\left[\ln\left(10x^3 - 8x^2 + 4x\right)\right] = \frac{30x^2-16x+4}{10x^3-8x^2+4x} = \frac{15x^2-8x+2}{5x^3-4x^2+2x}$

9. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2}{\sin(x)}}\right)\right] =$

Remark: We can compute this derivative directly, in it's current form, but it would be much easier to use the *algebraic properties* of natural logarithms to simplify the expression first.

$$\frac{d}{dx}\left[\ln\left[\left(\frac{x^2}{\sin(x)}\right)^{\frac{1}{2}}\right]\right] = \underbrace{\frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{x^2}{\sin\left(x\right)}\right)\right]}_{\ln(a^n) = n\ln(a)} = \underbrace{\frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(x^2\right) - \ln\left(\sin\left(x\right)\right)\right)\right]}_{\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)} = \frac{1}{2}\frac{d}{dx}\left[\ln\left(x^2\right) - \ln\left(\sin\left(x\right)\right)\right]$$

NOW we're ready to compute the derivative!

$$\frac{d}{dx}\left[\ln\left[\left(\frac{x^2}{\sin(x)}\right)^{\frac{1}{2}}\right]\right] = \frac{1}{2}\frac{d}{dx}\left[\ln\left(x^2\right) - \ln\left(\sin\left(x\right)\right)\right] = \frac{1}{2}\left[\frac{1}{x^2}\left(2x\right) - \frac{1}{\sin(x)}\cos\left(x\right)\right] = \frac{1}{2}\left(\frac{2x}{x^2} - \frac{\cos(x)}{\sin(x)}\right) = \frac{1}{x} - \frac{1}{2}\cot\left(x\right)$$

i.e., $\frac{d}{dx}\left[\ln\left[\left(\frac{x^2}{\sin(x)}\right)^{\frac{1}{2}}\right]\right] = \frac{1}{2}\left(\frac{2x}{x^2} - \frac{\cos(x)}{\sin(x)}\right) = \frac{1}{x} - \frac{1}{2}\cot\left(x\right)$

10. Compute: $\int_{x=-1}^{x=1} \left(x^5+1\right)^3 x^4\, dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(x^5+1\right)^3$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(x^5+1\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(x^5+1\right)}_{\text{function}} - - - - - \rightarrow \underbrace{x^4}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(x^5+1\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$
\begin{aligned}
u &= x^5+1 \\
\Rightarrow \frac{du}{dx} &= 5x^4 \\
\Rightarrow du &= 5x^4\, dx \\
\Rightarrow \frac{1}{5}du &= x^4\, dx
\end{aligned}
$$

When $x=-1$, $u = x^5+1 = (-1)^5+1 = 0$
When $x=1$, $u = x^5+1 = (1)^5+1 = 2$

3. Analyze in terms of u and du

$$\int_{x=-1}^{x=1} \underbrace{\left(x^5+1\right)^3}_{u^3} \underbrace{x^4\, dx}_{\frac{1}{5}du} = \int_{u=0}^{u=2} u^3 \cdot \frac{1}{5}du = \frac{1}{5}\int_{u=0}^{u=2} u^3 du$$

Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$\frac{1}{5}\int_{u=0}^{u=2} u^3 du = \frac{1}{5}\left[\frac{u^4}{4}\right]_{u=0}^{u=2} = \frac{1}{20}\left[u^4\right]_{u=0}^{u=2} = \underbrace{\frac{1}{20}(2)^4}_{F(2)} - \underbrace{\frac{1}{20}(0)^4}_{F(0)} = \frac{4}{5}$$

i.e., $\int_{x=-1}^{x=1} \left(x^5+1\right)^3 x^4\, dx = \frac{4}{5}$