MTH 1126 - Test #2 (9am Class) - Solutions

Spring 2024

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Instructions. Show CLEARLY how you arrive at your answers.

1. Use the "$f - g$" method to compute the area bounded by the graphs of $f(x) = x^2 - 4$ and $g(x) = -x - 2$.

   First, graph the functions and find the points of intersection.

   $y = x^2 - 4 = -x - 2$

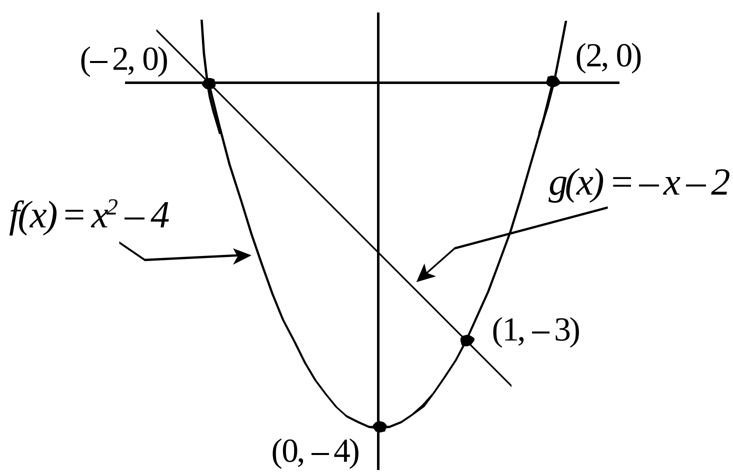
   $\Rightarrow x^2 + x - 4 = -2$

   $\Rightarrow x^2 + x - 2 = 0$

   $\Rightarrow (x + 2)(x - 1) = 0$

   $x = -2;\ x = 1$

   Points of intersection are $(-2, 0)$ and $(1, -3)$.



   Observe: The bounded region spans the interval $[-2, 1]$ on the x-axis.

   Also: Over the interval spanned by the bounded region, $g(x) \geq f(x)$.

   (Continued)

Hence, the area is given by:

$A = \int_{x=-2}^{x=1} (g(x) - f(x))\, dx = \int_{-2}^{1} [(-x-2) - (x^2-4)]\, dx = \int_{-2}^{1} (-x^2 - x + 2)\, dx$

$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1} = \left(-\frac{(1)^3}{3} - \frac{(1)^2}{2} + 2(1)\right) - \left(-\frac{(-2)^3}{3} - \frac{(-2)^2}{2} + 2(-2)\right)$

$= \frac{9}{2}$

i.e., bounded area $= \frac{9}{2}$

2. Find the area bounded by the graphs of $f(x) = x^2 - 4$ and $g(x) = 2x - 1$. (Partition the appropriate interval, sketch the i$^{\text{th}}$ rectangle, build the Riemann Sum, derive the appropriate integral.)

Graph the functions and find the points of intersection.

To find the points of intersection, set the y-coordinates equal to one another and solve for x.

$y = x^2 - 4 = 2x - 1$

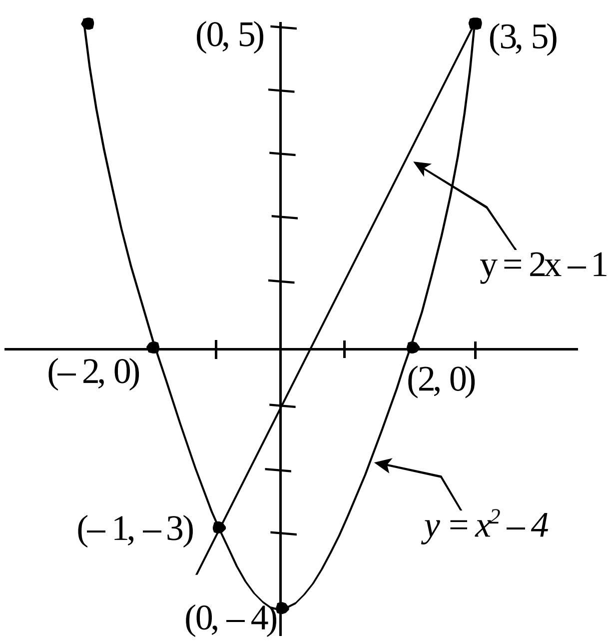
$\Rightarrow x^2 - 2x - 4 = -1$

$\Rightarrow x^2 - 2x - 3 = 0$

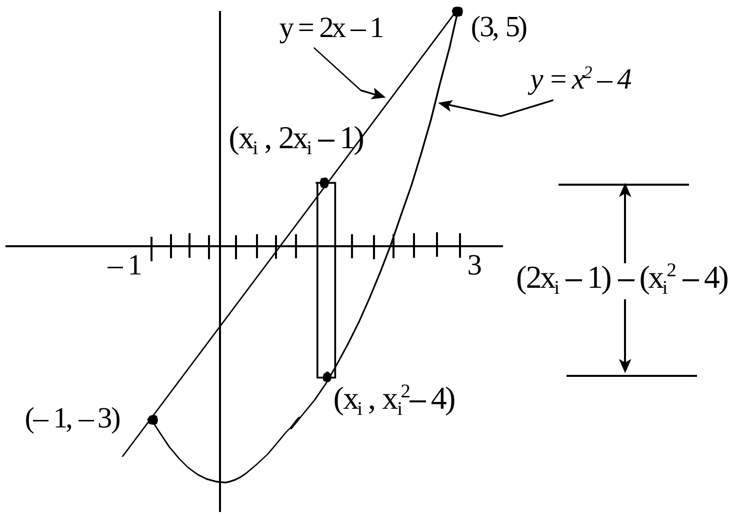
$\Rightarrow (x+1)(x-3) = 0$

$\Rightarrow x = -1$; and $x = 3$.

Points of intersection: $(-1, -3)$ and $(3, 5)$.



Inscribe a rectangle of width $\triangle x$ between the two graphs.



The rectangles span the interval $[-1,3]$ on the x-axis, so we will partition that interval into sub-intervals of length Δx.

The area of the i^{th}. rectangle is $\underbrace{\left((2x_i-1)-\left(x_i^2-4\right)\right)}_{\text{height}}\cdot\underbrace{\Delta x}_{\text{width}}=\left(-x_i^2+2x_i+3\right)\Delta x$

To approximate the area of the bounded region, we add the areas of the rectangles:

$A\approx\sum_{i=1}^{n}\left(-x_i^2+2x_i+3\right)\Delta x$

To get the exact area, we let $\Delta x\to 0$.

$A=\lim_{\Delta x\to 0}\sum_{i=1}^{n}\left(-x_i^2+2x_i+3\right)\Delta x=\int_{-1}^{3}\left(-x^2+2x+3\right)dx=\left[-\frac{1}{3}x^3+x^2+3x\right]_{-1}^{3}$

$=\left(-\frac{1}{3}(3)^3+(3)^2+3(3)\right)-\left(-\frac{1}{3}(-1)^3+(-1)^2+3(-1)\right)=\frac{32}{3}$

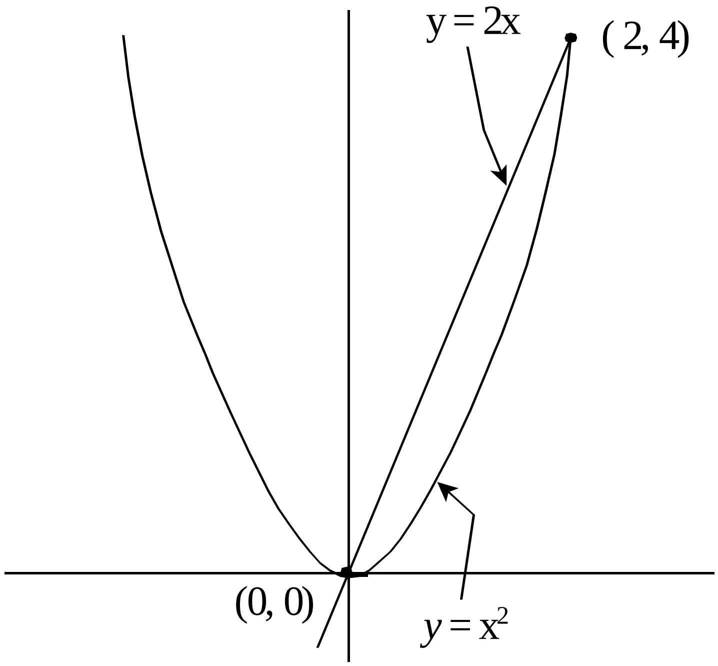
i.e., bounded area $=\frac{32}{3}$

3. Use the "shell method" to compute the volume of the solid of revolution generated by revolving the region bounded by the graphs of $y = 2x$, and $y = x^2$, about the y-axis. (For your information: the equation of the y-axis is $x = 0$.)

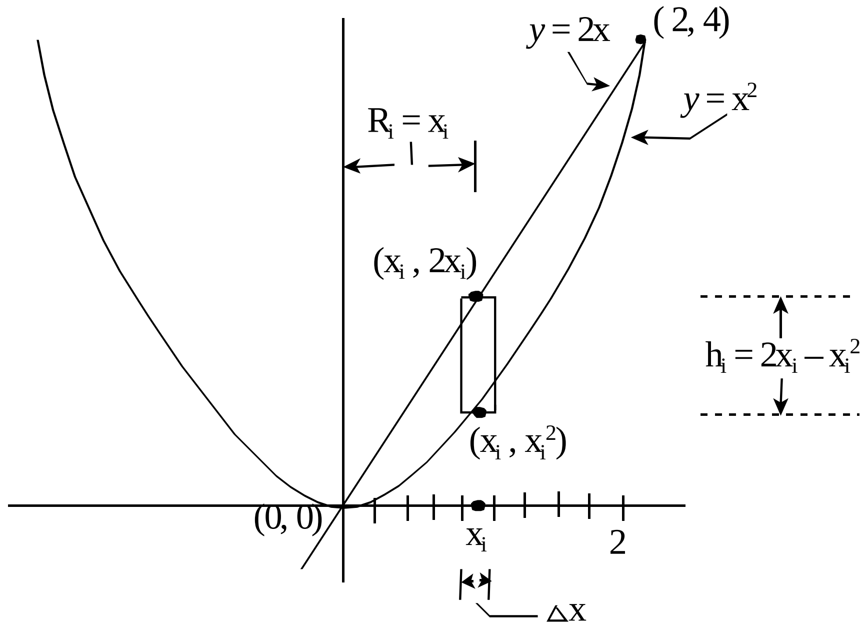
   Use the "five step method" (partition the interval, sketch the i$^{\text{th}}$ rectangle, form the sum, take the limit)

   Use the "five step method" (partition the interval, sketch the i$^{\text{th}}$ rectangle, form the sum, take the limit)

   i) First, we'll graph the bounded region.



ii) Next, we sketch a rectangle of width Δx parallel ("shell-parallel") to the axis of revolution, and we partition the interval spanned by the rectangles.



iii) Revolve the i[th] rectangle about the axis of revolution and compute the volume of the i[th] shell, Vol_i

$$Vol_i = 2\pi R_i h_i \Delta x = 2\pi x_i \left(2x_i - x_i^2\right) \Delta x = 2\pi \left(2x_i^2 - x_i^3\right) \Delta x$$

iv) Approximate the volume of the solid by adding up the volumes of the shells

$$Vol \approx \sum_{i=1}^{n} 2\pi \left(2x_i^2 - x_i^3\right) \Delta x$$

v) Let $\Delta x \to 0$

$$Vol = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 2\pi \left(2x_i^2 - x_i^3\right) \Delta x = \int_{x=0}^{x=2} 2\pi \left(2x^2 - x^3\right) dx = 2\pi \left[\frac{2}{3}x^3 - \frac{x^4}{4}\right]_{x=0}^{x=2}$$

$$= 2\pi \left[\left(\frac{2}{3}(2)^3 - \frac{(2)^4}{4}\right) - \left(\frac{2}{3}(0)^3 - \frac{(0)^4}{4}\right)\right] = \frac{8}{3}\pi$$

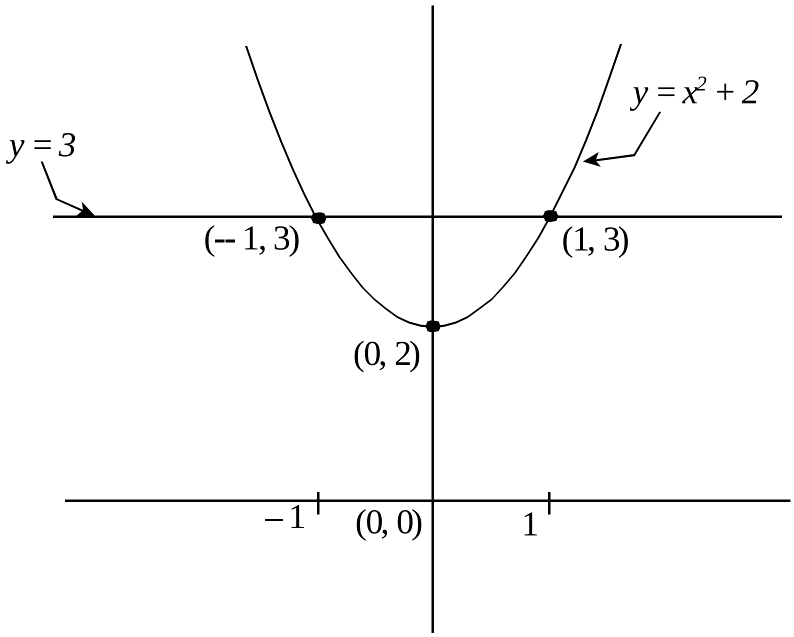
$$\boxed{Vol = \frac{8}{3}\pi}$$

4. Use the “disc method” to compute the volume of the solid of revolution generated by revolving the region described below about the x-axis.

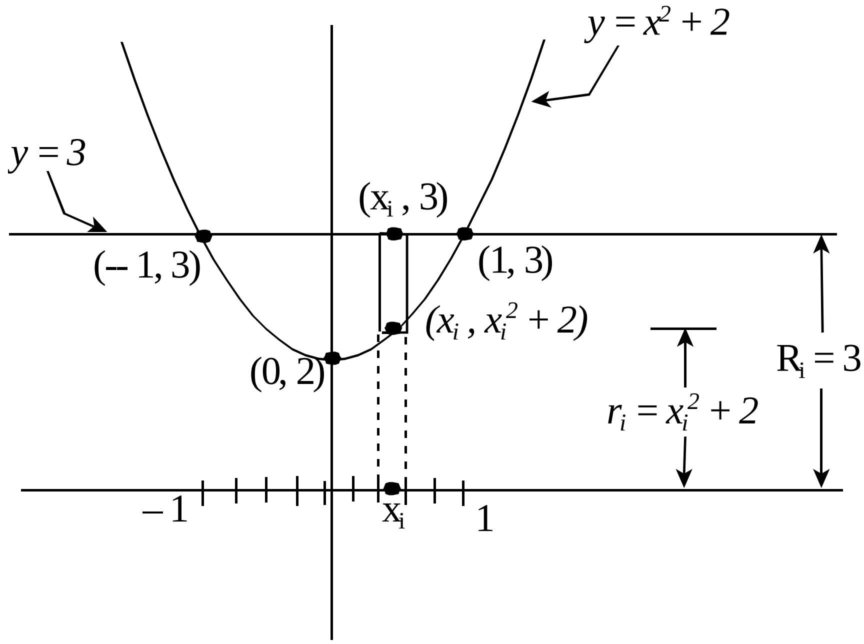
The region lies above the x-axis and is bounded by the graph $y = x^2 + 2$ and $y = 3$.

Use the “five step method” (partition the interval, sketch the i^{th} rectangle, form the sum, take the limit)

i) First, we’ll graph the bounded region.



ii) Next, we sketch a rectangle of width Δx perpendicular (perpen-"disc"-ular) to the axis of revolution, and we partition the interval spanned by the rectangles.



iii) Revolve the i$^{\text{th}}$ rectangle about the axis of revolution and compute the volume of the i$^{\text{th}}$ disc (or i$^{\text{th}}$ washer), Vol_i

Vol_i = Vol i$^{\text{th}}$ large disc $-$ Vol i$^{\text{th}}$ hole $= \pi R_i^2 \Delta x - \pi r_i^2 \Delta x = \pi (3)^2 \Delta x - \pi (x_i^2 + 2)^2 \Delta x$

$= \pi 9 \Delta x - \pi (x_i^4 + 4x_i^2 + 4) \Delta x = \pi (5 - x_i^4 - 4x_i^2) \Delta x$

iv) Approximate the volume of the solid by adding up the volumes of the discs (washers)

$Vol \approx \sum_{i=1}^{n} \pi (5 - x_i^4 - 4x_i^2) \Delta x$

v) Let $\Delta x \to 0$

$Vol = \lim_{\Delta x \to 0} \sum_{i=1}^{n} \pi (5 - x_i^4 - 4x_i^2) \Delta x = \int_{x=-1}^{x=1} \pi (5 - x^4 - 4x^2) \, dx$

$= \pi \left[5x - \frac{1}{5}x^5 - \frac{4}{3}x^3\right]_{x=-1}^{x=1} = \pi \left[5(1) - \frac{1}{5}(1)^5 - \frac{4}{3}(1)^3\right] - \pi \left[5(-1) - \frac{1}{5}(-1)^5 - \frac{4}{3}(-1)^3\right]$

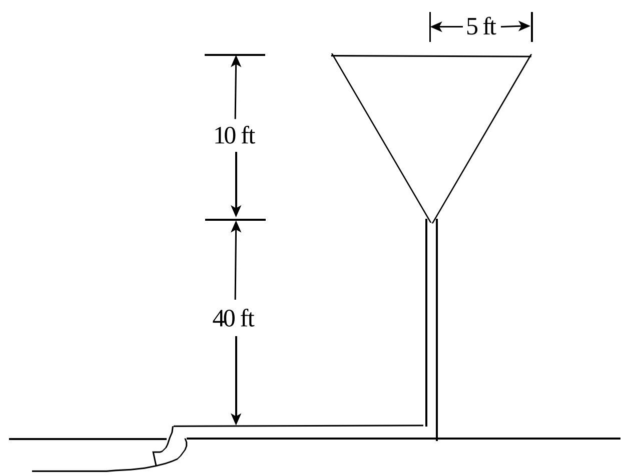
$= \pi \left[\left(\frac{52}{15}\right)\right] - \pi \left[\left(-\frac{52}{15}\right)\right] = \frac{104}{15}\pi$

$$\boxed{Vol = \frac{104}{15}\pi}$$

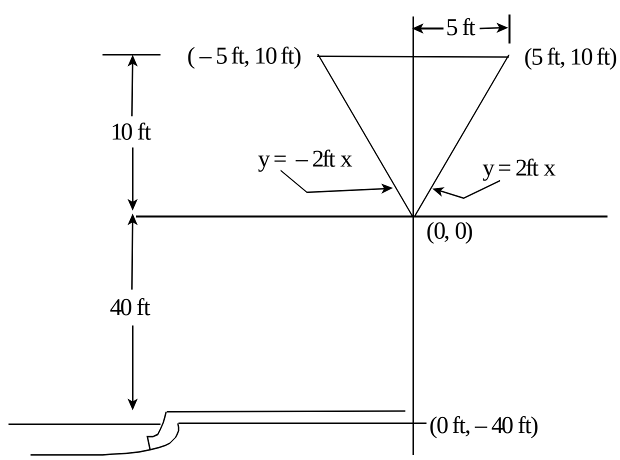
From exercises 5 and 6, select one.

5. Compute the work done in filling the reservoir of a water tower, though a hole in the bottom of the reservoir. The reservoir is a "cone-shaped" tank of height 10 ft and radius 5 ft at the top. The base of the reservoir is 40 ft above the level of the pond from which the water is pumped. (Assume that water weighs $\rho = 100\frac{\text{lbs}}{\text{ft}^3}$)

   The "layout" is depicted below:



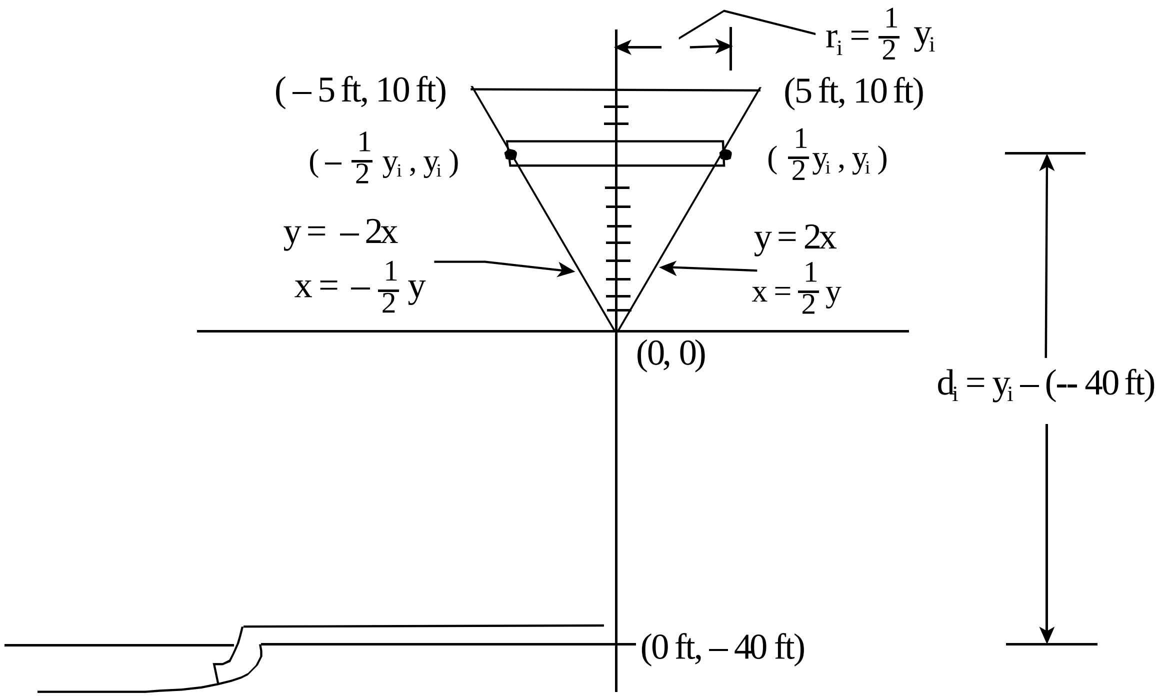
   For convenience, we'll situate the reservoir so that the vertex is at the origin.



   We'll partition the water into "slices" of thickness $\triangle x$ and compute the work done pumping the i^{th} slice from ground level to it's final height.

   Our assumption is that the i^{th} slice is so thin that every molecule within the slice is approximately the same distance from ground level.

   Thus, the distance over which the work in done in pumping the i^{th} slice to its final height will be constant.



We compute W_i the work done pumping the i^{th} slice to its final height.

$W_i = F_i \cdot d_i = w_i \cdot d_i$ where w_i is the weight of the i^{th} slice of water

$$w_i = \left(\text{volume of } i^{th} \text{ slice}\right) \cdot (\text{weight per unit volume})$$

$$= (\text{cross-sectional area } \cdot \text{ thickness}) \cdot \left(\rho = 100 \tfrac{\text{1bs}}{\text{ft}^3}\right)$$

$$= (\pi r_i^2 \cdot \triangle y) \cdot (100 \tfrac{\text{1bs}}{\text{ft}^3}) = 100 \tfrac{\text{1bs}}{\text{ft}^3} \pi r_i^2 \cdot \triangle y = 100 \tfrac{\text{1bs}}{\text{ft}^3} \pi \left(\tfrac{1}{2} y_i\right)^2 \cdot \triangle y$$

$$= 100 \tfrac{\text{1bs}}{\text{ft}^3} \pi \left(\tfrac{1}{4} y_i^2\right) \cdot \triangle y = 25 \tfrac{\text{1bs}}{\text{ft}^3} \pi y_i^2 \cdot \triangle y$$

i.e., $w_i = 25 \frac{\text{1bs}}{\text{ft}^3} \pi y_i^2 \cdot \triangle y$

Thus, $W_i = F_i \cdot d_i = w_i \cdot d_i = \left(25 \frac{\text{1bs}}{\text{ft}^3} \pi y_i^2 \cdot \triangle y\right) (y_i - (-40 \text{ ft}))$

$$= \left(25 \tfrac{\text{1bs}}{\text{ft}^3} \pi y_i^2 \cdot \triangle y\right) (y_i + 40 \text{ ft}) = 25 \tfrac{\text{1bs}}{\text{ft}^3} \pi y_i^2 \, (y_i + 40 \text{ ft}) \triangle y$$

$$= 25 \tfrac{\text{1bs}}{\text{ft}^3} \pi \, (y_i^3 + 40 \text{ ft} y_i^2) \triangle y$$

i.e., $W_i = 25 \frac{\text{1bs}}{\text{ft}^3} \pi \, (y_i^3 + 40 \text{ ft} y_i^2) \triangle y$

The Total Work Done in filling the reservoir W_T is given by:

$$W_T \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} 25\frac{\text{1bs}}{\text{ft}^3}\pi\left(y_i^3 + 40\text{ ft}y_i^2\right)\triangle y$$

To get the exact work done, we let $\triangle y \to 0$

$$W_T = \lim_{\triangle y \to 0}\sum_{i=1}^{n} W_i = \lim_{\triangle y \to 0}\sum_{i=1}^{n} 25\frac{\text{1bs}}{\text{ft}^3}\pi\left(y^3 + 40\text{ ft}y^2\right)\triangle y$$

$$= \int_{y=0\text{ft}}^{y=10\text{ft}} 25\frac{\text{1bs}}{\text{ft}^3}\pi\left(y^3 + 40\text{ ft}y^2\right)dy = 25\frac{\text{1bs}}{\text{ft}^3}\pi\left[\frac{y^4}{4} + 40\text{ ft}\frac{y^3}{3}\right]_{0\text{ft}}^{10\text{ft}}$$

$$= 25\frac{\text{1bs}}{\text{ft}^3}\pi\left[\frac{(10\text{ft})^4}{4} + 40\text{ ft}\frac{(10\text{ft})^3}{3}\right] - 25\frac{\text{1bs}}{\text{ft}^3}\pi\left[\frac{(0\text{ft})^4}{4} + 40\text{ ft}\frac{(0\text{ft})^3}{3}\right]$$

$$= 25\frac{\text{1bs}}{\text{ft}^3}\pi\left[2,500\text{ft}^4 + \frac{40,000}{3}\text{ft}^4\right] = 25\frac{\text{1bs}}{\text{ft}^3}\pi\left[\frac{47,500}{3}\text{ft}^4\right] = \frac{1187500}{3}\pi\text{ lb ft}$$

i.e., $W = \frac{1,187,500}{3}\pi$ lb ft

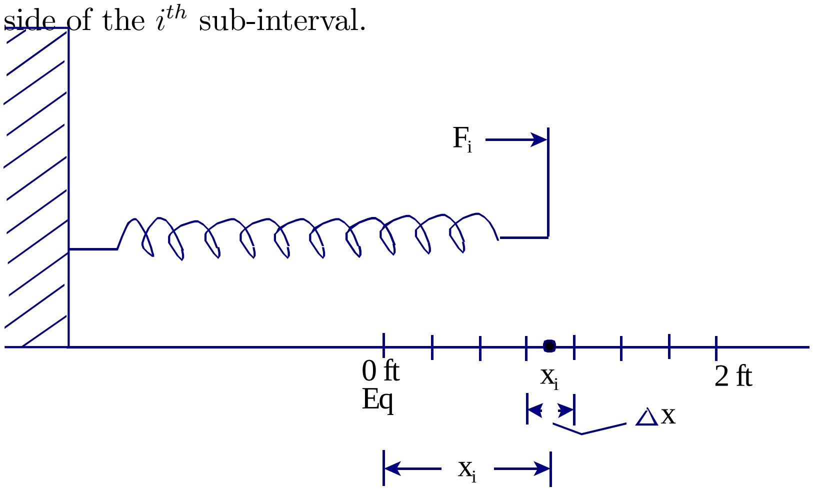
6. 10 lb of force is required to stretch a spring 1ft past the point of equilibrium. Compute the work done in stretching the spring a distance of 2 ft past the point of equilibrium.

First, find the spring constant, k, using the values $F = 10$ lb and $x = 1$ ft

From Hooke's Law, $F = kx$, we have $k = \frac{F}{x} = \frac{10 \text{ lb}}{1 \text{ ft}} = 10 \ \frac{\text{lb}}{\text{ft}}$

Hence, we have: $F = 10\frac{\text{lb}}{\text{ft}}x$

Next, partition the interval, over which the work is to be performed, and compute W_i, the work done stretching the spring from one side of the i^{th} sub-interval to the other side of the i^{th} sub-interval.



$W_i = F_i d_i$

Here, $d_i = \Delta x$

$F_i = kx_i = 10\frac{\text{lb}}{\text{ft}}x_i$

Hence, $W_i = 10\frac{\text{lb}}{\text{ft}}x_i\triangle x$

The total work, W_T, is approximately the sum of the work done stretching the spring across each sub-interval.

$W_T \approx \sum_{i=1}^{n} 10\frac{\text{lb}}{\text{ft}}x_i\triangle x$

$W_T = \lim_{\Delta x \to 0} \sum_{i=1}^{n} 10\frac{\text{lb}}{\text{ft}}x_i\triangle x = \int_{0 \text{ ft}}^{2 \text{ ft}} 10\frac{\text{lb}}{\text{ft}}x\,dx = 10\frac{\text{lb}}{\text{ft}}\int_{0 \text{ ft}}^{2 \text{ ft}} x\,dx = 10\frac{\text{lb}}{\text{ft}}\left[\frac{x^2}{2}\right]_{0 \text{ ft}}^{2 \text{ ft}}$

$= 10\frac{\text{lb}}{\text{ft}}\left[\left(\frac{(2 \text{ ft})^2}{2}\right) - \left(\frac{(0 \text{ ft})^2}{2}\right)\right] = 20 \text{ lb ft}$

$W_T = 20 \text{ lb ft}$