MTH 1126 - Test #2 - Solutions

SPRING 2008

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Instructions. Show CLEARLY how you arrive at your answers.

1. Compute the arclength of the graph of the function $f(x) = x^{\frac{3}{2}} + 6$ from the point $(1,7)$ and $(4,14)$.

   Arclength $= \int_{x=a}^{x=b} \sqrt{1 + (f'(x))^2}\,dx$

   We need $f'(x) = \frac{3}{2}x^{\frac{1}{2}}$

   Arclength $= \int_{x=a}^{x=b} \sqrt{1 + (f'(x))^2}\,dx = \int_{x=1}^{x=4} \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx = \int_{x=1}^{x=4} \sqrt{1 + \frac{9}{4}x}\,dx$

   $$\boxed{\begin{aligned} \text{Let } u &= 1 + \tfrac{9}{4}x \\ \Rightarrow \tfrac{du}{dx} &= \tfrac{9}{4} \\ \Rightarrow du &= \tfrac{9}{4}dx \\ \Rightarrow \tfrac{4}{9}du &= dx \end{aligned}}$$

   When $x = 1$, $u = 1 + \frac{9}{4}x = \frac{13}{4}$

   When $x = 4$, $u = 1 + \frac{9}{4}x = 10$

   So Arclength $= \int_{x=1}^{x=4} \underbrace{\sqrt{1 + \frac{9}{4}x}}_{u^{\frac{1}{2}}}\,\underbrace{dx}_{\frac{4}{9}du} = \int_{u=\frac{13}{4}}^{u=10} u^{\frac{1}{2}}\frac{4}{9}du = \frac{4}{9}\int_{u=\frac{13}{4}}^{u=10} u^{\frac{1}{2}}du = \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=\frac{13}{4}}^{u=10}$

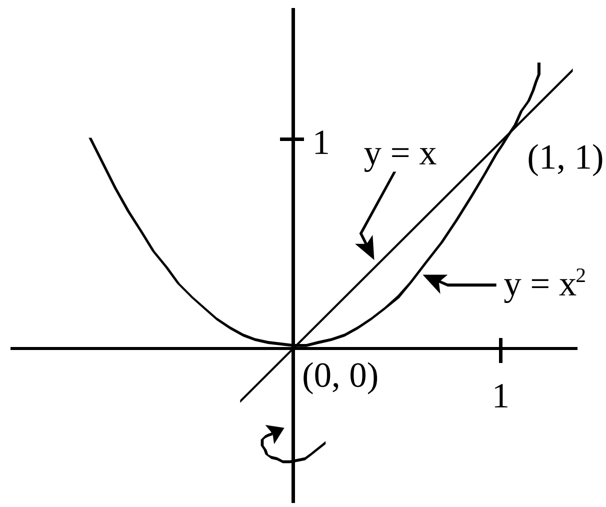
   $= \frac{8}{27}(10)^{\frac{3}{2}} - \frac{8}{27}\left(\frac{13}{4}\right)^{\frac{3}{2}}$

   i.e., Arclength $= \frac{8}{27}(10)^{\frac{3}{2}} - \frac{8}{27}\left(\frac{13}{4}\right)^{\frac{3}{2}}$

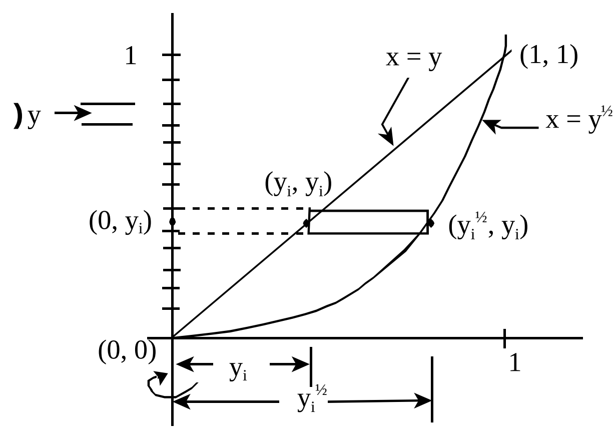
For problems 2 to 4, use the "five step method" (partition the interval, form the sum, take the limit)

2. Use the "disc method" to compute the volume of the solid of revolution generated by revolving the region bounded by the graphs $y = x$ and $y = x^2$ about the line y-axis.

   1. First, we graph the bounded region.



   2. Sketch a typical rectangle *perpendicular* to the axis of revolution, and partition the interval spanned by the rectangles.



   3. Compute the volume of the i^{th}donut.

   Vol i^{th} donut = vol i^{th} large disk − vol i^{th} hole

   Vol i^{th} donut $=\pi\left(y_i^{\frac{1}{2}}\right)^2\Delta y-\pi\left(y_i\right)^2\Delta y$

   Vol i^{th} donut $=\pi y_i\Delta y-\pi y_i^2\Delta y$

   Vol i^{th} donut $=\pi\left(y_i-y_i^2\right)\Delta y$

4. Approximate the volume of the solid of revolution by adding up the volumes of the donuts.

   Vol $\approx \sum_{i=1}^{n} \pi \left(y_i - y_i^2\right) \Delta y$

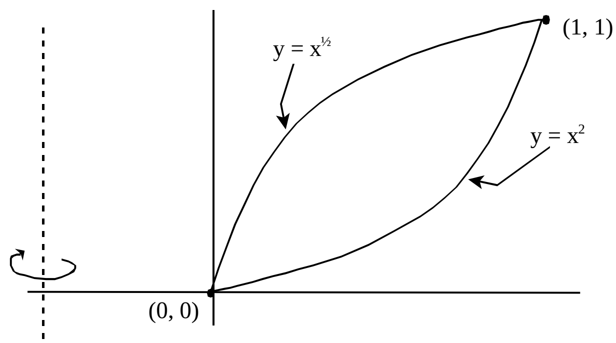
5. Let $\Delta y \to 0$

   Vol $= \int_{y=0}^{y=1} \pi \left(y - y^2\right) dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_{y=0}^{y=1} = \pi \left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right) - \left(\frac{0^2}{2} - \frac{0^3}{3}\right)\right] = \frac{\pi}{6}$

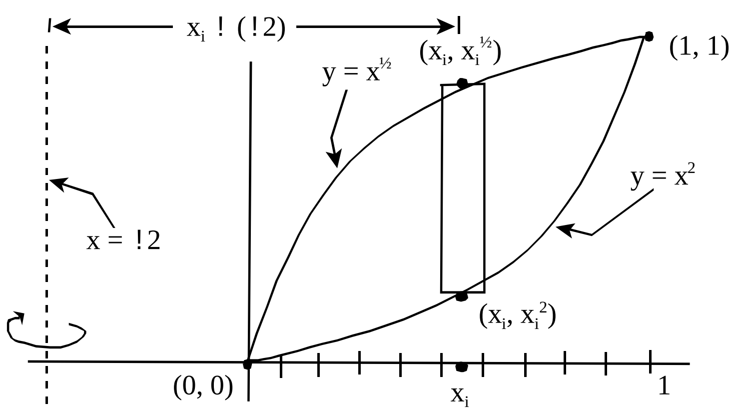
   i.e., Vol $= \frac{\pi}{6}$

3. Use the "shell method" to compute the volume of the solid of revolution generated by revolving the region bounded by the graph $f(x) = x^2$ and $f(x) = x^{\frac{1}{2}}$ about the line line $x = -2$.

   1. First, we graph the bounded region.



   2. Sketch a typical rectangle *parallel* to the axis of revolution, and partition the interval spanned by the rectangles.



   3. Compute the volume of the i^{th} shell

   $\text{Vol}_i = 2\pi \left(x_i - (-2)\right) \left(x_i^{\frac{1}{2}} - x_i^2\right) dx$

   $\text{Vol}_i = 2\pi \left(x_i + 2\right) \left(x_i^{\frac{1}{2}} - x_i^2\right) dx$

$\text{Vol}_i = 2\pi \left(-x_i^3 - 2x_i^2 + x_i^{\frac{3}{2}} + 2x_i^{\frac{1}{2}} \right) dx$

4. Approximate the volume of the solid by adding up the volumes of the shells.

   $\text{Vol} \approx \sum_{i=1}^{n} 2\pi \left(-x_i^3 - 2x_i^2 + x_i^{\frac{3}{2}} + 2x_i^{\frac{1}{2}} \right) dx$

5. Let $\Delta x \to 0$

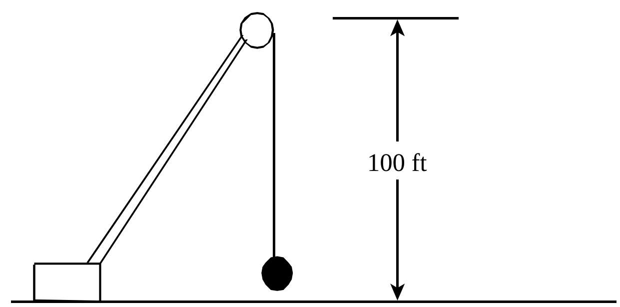
   $\text{Vol} = \int_{x=0}^{x=1} 2\pi \left(-x^3 - 2x^2 + x^{\frac{3}{2}} + 2x^{\frac{1}{2}} \right) dx = 2\pi \left[-\frac{x^4}{4} - \frac{2x^3}{3} + \frac{2x^{\frac{5}{2}}}{5} + \frac{4x^{\frac{3}{2}}}{3} \right]_{x=0}^{x=1}$

   $= 2\pi \left[\left(-\frac{(1)^4}{4} - \frac{2(1)^3}{3} + \frac{2(1)^{\frac{5}{2}}}{5} + \frac{4(1)^{\frac{3}{2}}}{3} \right) - \left(-\frac{(0)^4}{4} - \frac{2(0)^3}{3} + \frac{2(0)^{\frac{5}{2}}}{5} + \frac{4(0)^{\frac{3}{2}}}{3} \right) \right] = \frac{49\pi}{30}$

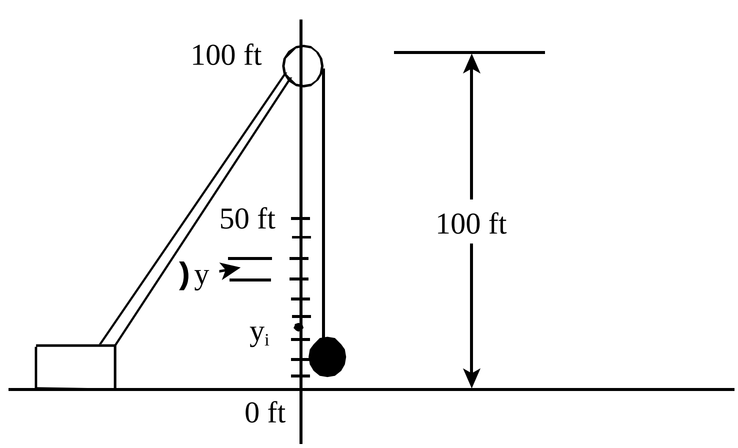
   i.e., $\text{Vol} = \frac{49\pi}{30}$.

4. A crane, which reaches to a height of 100 feet is used to raise a 200 lb wrecking ball. How much work is done raising the wrecking ball from ground level to a height of 50 feet, if the cable weighs 2 $\frac{\text{lb}}{\text{ft}}$?

   1. Draw a good picture



   2. Let W_i be the work done in raising the ball from the bottom to the top of the i^{th} sub-interval. Since the ball is raised to a height of 50 ft, the interval $y = 0$ ft to $y = 50$ ft will be partitioned into sub-intervals.



$$
\begin{aligned}
W_i &\approx F_i D_i \\
&\approx (\text{weight of ball} + \text{weight cable})\Delta y \\
&\approx ((200\text{ lb}) + \rho(\text{length of cable}))\Delta y \\
&\approx ((200\text{ lb}) + \rho(\text{length of cable}))\Delta y \\
&\approx \left((200\text{ lb}) + 2\tfrac{\text{lb}}{\text{ft}}\,(100\,\text{ft} \ - y_i)\right)\Delta y \\
&\approx \left(200\text{ lb} \ + 200\text{ lb} \ - 2\tfrac{\text{lb}}{\text{ft}}\ y_i\right)\Delta y \\
&\approx \left(400\text{ lb} \ - 2\tfrac{\text{lb}}{\text{ft}}\ y_i\right)\Delta y
\end{aligned}
$$

3. Approximate the total work done by adding up the work done over each sub-interval.

   $W_{\text{total}} \approx \sum_{i=1}^{n} \left(400\text{ lb} \ - 2\frac{\text{lb}}{\text{ft}}\ y_i\right)\Delta y$

4. Let $\Delta y \to 0$

   $W_{\text{total}} = \int_{y=0\text{ ft}}^{y=50\text{ ft}} \left(400\text{ lb} \ - 2\frac{\text{lb}}{\text{ft}}\ y\right)dy = \left[400y\text{ lb} - y^2\frac{\text{lb}}{\text{ft}}\right]_{y=0\text{ft}}^{y=50\text{ ft}}$

   $= \left[\left(400\,(50\text{ ft})\text{ lb} - (50\text{ ft})^2\,\frac{\text{lb}}{\text{ft}}\right) - \left(400\,(0\text{ ft})\text{ lb} - (0\text{ ft})^2\,\frac{\text{lb}}{\text{ft}}\right)\right]$

   $= (20000\text{ ft lb} \ - 2500\text{ ft lb}) - (0\text{ ft lb}) = 17,500\text{ ft lb}$

   i.e., $W_{\text{total}} = 17,500\text{ ft lb}$