MTH 1125 2pm Class - Test #4 - Solutions

Fall 2021

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Show CLEARLY how you arrive at your answers!

1. **Compute:** $\int \left(24x^3 + 33x^2 - 14x + 7\sqrt{x} + 12\right) dx =$

$$\int \left(24x^3 + 33x^2 - 14x + 7\sqrt{x} + 12\right) dx = \int \left(24x^3 + 33x^2 - 14x + 7x^{\frac{1}{2}} + 12\right) dx$$

$$= 24\left[\tfrac{x^4}{4}\right] + 33\left[\tfrac{x^3}{3}\right] - 14\left[\tfrac{x^2}{2}\right] + 7\left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right] + 12x + C = 6x^4 + 11x^3 - 7x^2 + \tfrac{14}{3}x^{\frac{3}{2}} + 12x + C$$

$$\boxed{\text{i.e., } \int \left(24x^3 + 33x^2 - 14x + 7\sqrt{x} + 12\right) dx = 6x^4 + 11x^3 - 7x^2 + \tfrac{14}{3}x^{\frac{3}{2}} + 12x + C}$$

2. **Compute:** $\int (12x^2+48x+8)^4 (x+2)\,dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $(12x^2+48x+8)^4$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = (12x^2+48x+8)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{(12x^2+48x+8)}_{\text{function}} - - - - \rightarrow \underbrace{(x+2)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = (12x^2+48x+8)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 12x^2+48x+8 \\ \Rightarrow \frac{du}{dx} &= 24x+48 \\ \Rightarrow du &= (24x+48)\,dx \\ \Rightarrow \frac{1}{24}du &= (x+2)\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{(12x^2+48x+8)^4}_{u^4}\underbrace{(x+2)\,dx}_{\frac{1}{24}du} = \int u^4 \tfrac{1}{24}du = \tfrac{1}{24}\int u^4 du$$

4. Integrate (in terms of u).

$$\tfrac{1}{24}\int u^4 du = \tfrac{1}{24}\left[\tfrac{u^5}{5}\right] + C = \tfrac{1}{120}u^5 + C$$

5. Re-express in terms of the original variable, x.

$$\int (12x^2+48x+8)^4 (x+2)\,dx = \underbrace{\frac{1}{120}\left(12x^2+48x+8\right)^5 + C}_{\frac{1}{120}u^5+C}$$

i.e., $\int \left(12x^2+48x+8\right)^4 (x+2)\,dx = \frac{1}{120}\left(12x^2+48x+8\right)^5 + C$

3. **Compute:** $\int \left(2\cos(x) - 5\sec^2(x) + 4\sec(x)\tan(x)\right) dx =$

$\int \left(2\cos(x) - 5\sec^2(x) + 4\sec(x)\tan(x)\right) dx = 2\sin(x) - 5\tan(x) + 4\sec(x) + C$

i.e., $\int \left(2\cos(x) - 5\sec^2(x) + 4\sec(x)\tan(x)\right) dx = 2\sin(x) - 5\tan(x) + 4\sec(x) + C$

4. **Compute:** $\int \sin\left(4x^3+6x+3\right)\left(4x^2+2\right)dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\sin\left(4x^3+6x+3\right)$

   $\nearrow$ outer $\nwarrow$ inner

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(4x^3+6x+3\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(4x^3+6x+3\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(4x^2+2\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(4x^3+6x+3\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 4x^3+6x+3 \\ \Rightarrow \tfrac{du}{dx} &= 12x^2+6 \\ \Rightarrow du &= \left(12x^2+6\right)dx \\ \Rightarrow \tfrac{1}{3}du &= \left(4x^2+2\right)dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\sin\left(4x^3+6x+3\right)}_{\sin(u)}\underbrace{\left(4x^2+2\right)dx}_{\frac{1}{3}du} = \int \sin(u)\ \tfrac{1}{3}du = \tfrac{1}{3}\int \sin(u)\,du$$

4. Integrate (in terms of u).

$$\tfrac{1}{3}\int \sin(u)\,du = \tfrac{1}{3}\left[-\cos(u)\right]+C = -\tfrac{1}{3}\cos(u)+C$$

5. Re-express in terms of the original variable x.

$$\int \sin\left(4x^3+6x+3\right)\left(4x^2+2\right)dx = \underbrace{-\frac{1}{3}\cos\left(4x^3+6x+3\right)+C}_{-\frac{1}{3}\cos(u)+C}$$

i.e., $\int \sin\left(4x^3+6x+3\right)\left(4x^2+2\right)dx = -\frac{1}{3}\cos\left(4x^3+6x+3\right)+C$

5. **Compute:** $\int_{-1}^{1}\left(4x^3+6x^2+2\right)dx =$

$$\int_{-1}^{1}\underbrace{\left(4x^3+6x^2+2\right)}_{f(x)}dx = \underbrace{\left[4\frac{x^4}{4}+6\frac{x^3}{3}+2x\right]_{-1}^{1}}_{F(x)} = \underbrace{\left[x^4+2x^3+2x\right]_{-1}^{1}}$$

$$= \underbrace{\left[(1)^4+2\,(1)^3+2\,(1)\right]}_{F(1)} - \underbrace{\left[(-1)^4+2\,(-1)^3+2\,(-1)\right]}_{F(-1)}$$

$$= 5-(-3) = 8$$

i.e., $\int_{-1}^{1}\left(4x^3+6x^2+2\right)dx = 8$

6. **Compute:** $\int_{-1}^{1} \left(x^3+1\right)^3 x^2 \; dx =$ (The answer may not be a whole number)

1. Is u-sub appropriate?

   a. Is there a composite function?

      Yes! $\left(x^3+1\right)^3$ (A function raised to a power is *always* a composite function!)

      Let $u =$ the "inner" of the composite function

      $\Rightarrow u = (x^3+1)$

   b. Is there an (approximate) function/derivative pair?

      Yes! $\underbrace{\left(x^3+1\right)}_{\text{function}} - - - - \rightarrow \underbrace{x^2}_{\text{deriv}}$

      Let $u =$ the "function" of the function/deriv pair

      $\Rightarrow u = (x^3+1)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

      (i.e., do criteria **a** and **b** suggest the same choice of u ?)

      Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= x^3+1 \\ \Rightarrow \tfrac{du}{dx} &= 3x^2 \\ \Rightarrow du &= 3x^2 \; dx \\ \Rightarrow \tfrac{1}{3} du &= x^2 \; dx \end{aligned}$$

When $x=-1$, $u = x^3+1 = (-1)^3+1 = 0$
When $x=1$, $u = x^3+1 = (1)^3+1 = 2$

3. Analyze in terms of u and du

$$\int_{x=-1}^{x=1} \underbrace{\left(x^3+1\right)^3}_{u^3} \underbrace{x^2 \; dx}_{\frac{1}{3}du} = \int_{u=0}^{u=2} u^3 \cdot \tfrac{1}{3} du = \tfrac{1}{3} \int_{u=0}^{u=2} u^3 du$$

Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$\frac{1}{3}\int_{u=0}^{u=2} u^3 du = \frac{1}{3}\left[\frac{u^4}{4}\right]_{u=0}^{u=2} = \left[\frac{u^4}{12}\right]_{u=0}^{u=2} = \underbrace{\frac{(2)^4}{12}}_{F(2)} - \underbrace{\frac{(0)^4}{12}}_{F(0)} = \frac{16}{12} - \left(\frac{0}{12}\right) = \frac{4}{3}$$

i.e., $\int_{-1}^{1} \left(x^3+1\right)^3 x^2 \ dx = \frac{4}{3}$

7. **Compute:** $\frac{d}{dx}\left[\ln\left(6x^3+6x^2-2x\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(6x^3+6x^2-2x\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{6x^3+6x^2-2x}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(18x^2+12x-2\right)}_{g'(x)} = \frac{18x^2+12x-2}{6x^3+6x^2-2x} = \frac{9x^2+6x-1}{3x^3+3x^2-x}$$

i.e., $\frac{d}{dx}\left[\ln\left(6x^3+6x^2-2x\right)\right] = \frac{18x^2+12x-2}{6x^3+6x^2-2x} = \frac{9x^2+6x-1}{3x^3+3x^2-x}$

Extra! (Wow! - 5 pts. Can you believe it?) **Compute:** $\int \frac{3x+2}{9x^2+12x+15} dx =$

$$\int \frac{3x+2}{9x^2+12x+15} dx \underbrace{=}_{\text{re-write}} \int \frac{1}{9x^2+12x+15} \, (3x+2) \, dx$$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{9x^2+12x+15}$ is the same as $\left(9x^2+12x+15\right)^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = 9x^2 + 12x + 15$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{(9x^2+12x+15)}_{\text{function}} - - - - \rightarrow \underbrace{(3x+2)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = 9x^2 + 12x + 15$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 9x^2 + 12x + 15 \\ \Rightarrow \frac{du}{dx} &= 18x + 12 \\ \Rightarrow du &= (18x+12)\,dx \\ \Rightarrow \frac{1}{6}du &= (3x+2)\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\underbrace{\int \frac{1}{9x^2+12x+15}}_{\frac{1}{u}} \underbrace{(3x+2)\,dx}_{\frac{1}{6}du} = \int \frac{1}{u}\frac{1}{6}du = \frac{1}{6}\int \frac{1}{u}du$$

4. Integrate (in terms of u).

   $\frac{1}{6}\int \frac{1}{u}du = \frac{1}{6}\ln|u| + C$

5. Re-express in terms of the original variable, x.

   $$\int \frac{3x+2}{9x^2+12x+15}dx = \underbrace{\frac{1}{6}\ln\left|9x^2+12x+15\right| + C}_{\frac{1}{6}\ln|u|+C}$$

   i.e., $\int \frac{3x+2}{9x^2+12x+15}dx = \frac{1}{6}\ln|9x^2+12x+15| + C$

Extra! (Wow! - 5 pts. Can you believe it?) **Compute:** $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{6x^4-4x^2+2}{8x^3-8x}}\right)\right] =$

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{6x^4-4x^2+2}{8x^3-8x}}\right)\right] \underset{\text{rewrite}}{=} \frac{d}{dx}\left[\ln\left[\left(\frac{6x^4-4x^2+2}{8x^3-8x}\right)^{\frac{1}{2}}\right]\right] \underset{\text{rewrite}}{=} \frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{6x^4-4x^2+2}{8x^3-8x}\right)\right]$$

$$\underset{\text{rewrite}}{=} \frac{1}{2}\frac{d}{dx}\left[\ln\left(\frac{6x^4-4x^2+2}{8x^3-8x}\right)\right] \underset{\text{rewrite}}{=} \frac{1}{2}\frac{d}{dx}\left[\ln\left(6x^4-4x^2+2\right) - \ln\left(8x^3-8x\right)\right]$$

$$\underset{\text{rewrite}}{=} \frac{1}{2}\left(\frac{1}{6x^4-4x^2+2}\left(24x^3-8x\right) - \frac{1}{8x^3-8x}\left(24x^2-8\right)\right) \underset{\text{rewrite}}{=} \frac{12x^3-4x}{6x^4-4x^2+2} - \frac{12x^2-4}{8x^3-8x}$$

$$\boxed{\frac{d}{dx}\left[\ln\left(\sqrt{\frac{6x^4-4x^2+2}{8x^3-8x}}\right)\right] = \frac{12x^3-4x}{6x^4-4x^2+2} - \frac{12x^2-4}{8x^3-8x} = \frac{6x^3-2x}{3x^4-2x^2+1} - \frac{3x^2-1}{2x^3-2x}}$$