Proofs Involving Sets #3 (Biconditional Statements) - Solutions

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Instructions. Prove the following.

1. $A \cap B = \emptyset \Leftrightarrow (B \cap A^c) = B$

   Proof. We must prove:

   i. $A \cap B = \emptyset \Rightarrow (B \cap A^c) = B$

   and

   ii. $(B \cap A^c) = B \Rightarrow A \cap B = \emptyset$

   i. $\boxed{A \cap B = \emptyset \Rightarrow (B \cap A^c) = B}$

   Let the hypothesis be given. (i.e., let $A \cap B = \emptyset$)

   We have to show that:

   a. $(B \cap A^c) \subseteq B$ (This is *always* true — $(B \cap$ *anything* $) \subseteq B$)

   and

   b. $B \subseteq (B \cap A^c)$

   Let $x \in B$

   $\Rightarrow x \notin A$ (Otherwise, if x *were* an element of A, then $x \in A \cap B$, contrary hypothesis.)

   i.e., $x \in B$ and $x \notin A$.

   $x \in B$ and $x \in A^c$

   $\Rightarrow x \in (B \cap A^c)$

   We have shown that $x \in B \Rightarrow x \in (B \cap A^c)$

   Therefore, $B \subseteq (B \cap A^c)$.

ii. $\boxed{(B \cap A^c) = B \Rightarrow A \cap B = \emptyset}$

Let the hypothesis be given. (i.e., let $(B \cap A^c) = B$)

We must show that $A \cap B = \emptyset$

To show this, we will show that x can not be an element of both A and B.

In turn, to show this, we will show that either:

a. $x \in A \Rightarrow x \notin B$

   or

b. $x \in B \Rightarrow x \notin A$

   I choose to show the latter, $x \in B \Rightarrow x \notin A$.

   Let $x \in B$.

   $\Rightarrow x \in (B \cap A^c)$ (because $B = (B \cap A^c)$) by hypothesis.)

   $\Rightarrow x \in B$ and $x \in A^c$

   $\Rightarrow x \in B$ and $x \notin A$

   In particular, $x \notin A$.

   We have shown that $x \in B \Rightarrow x \notin A$.

   Therefore, $A \cap B = \emptyset$ ■

2. $A \cap B = \emptyset \Leftrightarrow (A \cup B^c) = B^c$

Proof. We must prove:

i. $A \cap B = \emptyset \Rightarrow (A \cup B^c) = B^c$

and

ii. $(A \cup B^c) = B^c \Rightarrow A \cap B = \emptyset$

i. $\boxed{A \cap B = \emptyset \Rightarrow (A \cup B^c) = B^c}$

Let the hypothesis be given. (i.e., let $A \cap B = \emptyset$)

We have to show that:

a. $(A \cup B^c) \subseteq B^c$

and

b. $B^c \subseteq (A \cup B^c)$ (This is *always* true — $B^c \subseteq (B^c \cup$ *anything*).)

a. $(A \cup B^c) \subseteq B^c$

Let $x \in (A \cup B^c)$

$\Rightarrow x \in A$ or $x \in B^c$

$\Rightarrow x \notin B$ or $x \in B^c$ (Because $A \cap B = \emptyset$ by hypothesis)

$\Rightarrow x \in B^c$ or $x \in B^c$

$\Rightarrow x \in B^c$

i.e., $x \in (A \cup B^c) \Rightarrow x \in B^c$

Hence, $(A \cup B^c) \subseteq B^c$

ii. $\boxed{(A \cup B^c) = B^c \Rightarrow A \cap B = \emptyset}$

Let the hypothesis be given. (i.e., let $(A \cup B^c) = B^c$)

We must show that $A \cap B = \emptyset$

To show this, we will show that x can not be an element of both A and B.

In turn, to show this, we will show that either:

a. $x \in A \Rightarrow x \notin B$

   or

b. $x \in B \Rightarrow x \notin A$

   I choose to show the former, $x \in A \Rightarrow x \notin B$.

   Let $x \in A$.

   $\Rightarrow x \in (A \cup B^c)$ (because $A \subseteq (A \cup$ *anything* $))$

   $\Rightarrow x \in B^c$ (Because $(A \cup B^c) = B^c$ by hypothesis)

   $\Rightarrow x \notin B$

   We have shown that $x \in A \Rightarrow x \notin B$.

   Therefore, $A \cap B = \emptyset$ ■

3. $A \subseteq B \Leftrightarrow (B \cap C) \cup A = B \cap (C \cup A)$ for all sets A.

Proof. We must prove:

i. $A \subseteq B \Rightarrow (B \cap C) \cup A = B \cap (C \cup A)$ for all sets A.

and

ii. $(B \cap C) \cup A = B \cap (C \cup A)$ for all sets $A \Rightarrow A \subseteq B$

i. $\boxed{A \subseteq B \Rightarrow (B \cap C) \cup A = B \cap (C \cup A) \text{ for all sets } A.}$

Let the hypothesis be given. (i.e., let $A \subseteq B$)

We have to show that:

a. $(B \cap C) \cup A \subseteq B \cap (C \cup A)$

and

b. $B \cap (C \cup A) \subseteq (B \cap C) \cup A$

a. $(B \cap C) \cup A \subseteq B \cap (C \cup A)$

Let $x \in (B \cap C) \cup A$

$\Rightarrow x \in (B \cap C)$ or $x \in A$

$\Rightarrow x \in B$ and $x \in C$, or $x \in A$

$\Rightarrow x \in B$ or $x \in A$, and $x \in C$ or $x \in A$

$\Rightarrow x \in B$ or $x \in B$, and $x \in C$ or $x \in A$ (Because $A \subseteq B$) by hypothesis.)

$\Rightarrow x \in B$, and $x \in C$ or $x \in A$

$\Rightarrow x \in B$, and $x \in (C \cup A)$

$\Rightarrow x \in B \cap (C \cup A)$

i.e., $x \in (B \cap C) \cup A \Rightarrow x \in B \cap (C \cup A)$

Hence, $(B \cap C) \cup A \subseteq B \cap (C \cup A)$

b. $B \cap (C \cup A) \subseteq (B \cap C) \cup A$

Let $x \in B \cap (C \cup A)$

$\Rightarrow x \in B$ and $x \in (C \cup A)$

$\Rightarrow x \in B$ and, $x \in C$ or $x \in A$

$\Rightarrow x \in B$ and $x \in C$, or $x \in B$ and $x \in A$

$\Rightarrow x \in B \cap C$, or $x \in B \cap A$

$\Rightarrow x \in B \cap C$, or $x \in A$ (Because $(B \cap A) \subseteq A$ — always!)

$\Rightarrow x \in (B \cap C) \cup A$

i.e., $x \in B \cap (C \cup A) \Rightarrow x \in (B \cap C) \cup A$

Hence, $B \cap (C \cup A) \subseteq (B \cap C) \cup A$

ii. $(B \cap C) \cup A = B \cap (C \cup A)$ for all sets $A \Rightarrow A \subseteq B$

Let the hypothesis be given. (i.e., let $(B \cap C) \cup A = B \cap (C \cup A)$ for all sets A)

We must show that $A \cap B = \emptyset$

To show this, we will show that x can not be an element of both A and B.

In turn, to show this, we will show that either:

a. $x \in A \Rightarrow x \notin B$

or

b. $x \in B \Rightarrow x \notin A$

I choose to show the former, $x \in A \Rightarrow x \notin B$.

Let $x \in A$.

$\Rightarrow x \in (A \cup B^c)$ (because $A \subseteq (A \cup$ *anything* $))$

$\Rightarrow x \in B^c$ (Because $(A \cup B^c) = B^c$ by hypothesis)

$\Rightarrow x \notin B$

We have shown that $x \in A \Rightarrow x \notin B$.

Therefore, $A \cap B = \emptyset$ ■