Proofs Involving Functions 2a (Countability)

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1. $\sqrt{2}$ is an irrational number.

Proof. Suppose, for the sake of contradiction, that $\sqrt{2}$ is rational. Then there exist integers m and n, with $n \neq 0$, such that $\sqrt{2} = \frac{m}{n}$.

Without loss of generality, we can assume that m and n are relatively prime.*

$\Rightarrow \sqrt{2} = \frac{m}{n}$

$\Rightarrow 2 = \frac{m^2}{n^2}$

$\Rightarrow 2n^2 = m^2$

$\Rightarrow m^2$ is even

$\Rightarrow m$ is even.

$\Rightarrow \exists\ k \in \mathbf{Z}$ such that $m = 2k$.

Thus, $2n^2 = m^2 = (2k)^2 = 4k^2$

i.e., $2n^2 = 4k^2$

$\Rightarrow n^2 = 2k^2$

$\Rightarrow n^2$ is even

$\Rightarrow n$ is even

i.e., m and n are both even.

This contradicts the assumption that m and n are relatively prime.

Since the assumption that $\sqrt{2}$ is rational lead us to this contradiction, $\sqrt{2}$ must be irrational. ■

*If m and n are not relatively prime, then let d be the greatest common divisor of m and n. There exist relatively prime integers m_1 and n_1 such that $m = dm_1$ and $n = dn_1$. Thus we can write $\sqrt{2} = \frac{m}{n} = \frac{dm_1}{dn_1} = \frac{m_1}{n_1}$, and $\sqrt{2}$ is written as the quotient of relatively prime integers.

2. The set of positive rational numbers $\mathbf{Q}^{+}$ is denumerable.

   Consider the table of ordered pairs below:

$$
\begin{array}{cccccccccc}
(1,1) & \rightarrow & (1,2) & & (1,3) & \rightarrow & (1,4) & & (1,5) & \rightarrow \; \ldots \\
 & \swarrow & & \nearrow & & \swarrow & & \nearrow & & \\
(2,1) & & (2,2) & & (2,3) & & (2,4) & & (2,5) & \ldots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & & & \\
(3,1) & & (3,2) & & (3,3) & & (3,4) & & (3,5) & \ldots \\
 & \swarrow & & \nearrow & & & & & & \\
(4,1) & & (4,2) & & (4,3) & & (4,4) & & (4,5) & \ldots \\
\downarrow & \nearrow & & & & & & & & \\
(5,1) & & (5,2) & & (5,3) & & (5,4) & & (5,5) & \ldots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
$$

   If we consider the ordered pair (i, j) in the ith row and jth column to represent the quotient of integers $\frac{i}{j}$, then every positive rational number appears in the table at least once. (e.g., the rational number $\frac{m}{n}$ appears in the mth row and n^{th} column.)

   Furthermore, the arrows in the table induce an exhaustive **ordering** of the positive rational numbers as follows:

   $1, \frac{1}{2}, 2, 3, \frac{1}{3}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4, 5, \frac{1}{5}, \ldots$

   (Note that we have discarded repititions of rationals if they occur. e.g., we have discarded $(2, 2)$ because it is equaivalent to $(1, 1)$ which is already on our list.)

   Note also that since the positive rationals are **ordered,** they are in a one to one correspondence with the natural numbers.

   Hence, the positive rational numbers are denumerable. ■

3. The set of negative rational numbers $\mathbf{Q}^{-}$ is denumerable.

   Proof. The function $f : \mathbf{Q}^{+} \rightarrow \mathbf{Q}^{-}$ given by $f\left(\frac{m}{n}\right) = -\frac{m}{n}$ is clearly one to one and onto.

   For if $f(x_1) = f(x_2)$,

   Then $-x_1 = -x_2$

   $\Rightarrow x_1 = x_2,$ thus f is one to one.

   Also, given $y \in \mathbf{Q}^{-}$, we can choose $x \in \mathbf{Q}^{+}$, given by $x = -y$.

   This yields $f(x) = -x = -(-y) = y$.

   Thus, f is onto. ■

4. The union of a denumerable set and a finite set is denumerable (you can assume that the two sets are disjoint).

Proof. Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, b_3, \ldots\}$.

Then A is finite and B is denumerable.

Observe:

$$\begin{array}{ccccccccccccc} \mathbf{N} & = & \{ & 1, & 2, & 3, & \ldots, & k, & k+1, & k+2, & k+3, & \ldots & \} \\ f\downarrow & & & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & & \\ (A \cup B) & = & \{ & a_1, & a_2, & a_3, & \ldots, & a_k & b_1, & b_2, & b_3, & \ldots & \} \end{array}$$

Define $f : \mathbf{N} \to (A \cup B)$ by $f(n) = \begin{cases} a_n & \text{if } n \leq k \\ b_{n-k} & \text{if } n > k \end{cases}$

Clearly from the diagram above, f is one to one and onto. Hence, $(A \cup B)$ is denumerable. ■

5. The union of two (disjoint) denumerable sets is denumerable.

Proof. Let $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$

Observe:

$$\begin{array}{cccccccccc} \mathbf{N} & = & \{ & 1, & 2, & 3, & 4, & 5, & 6, & \ldots \ \} \\ f\downarrow & & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ (A \cup B) & = & \{ & a_1, & b_1, & a_2, & b_2, & a_3, & b_3, & \ldots \ \} \end{array}$$

Define $f : \mathbf{N} \to (A \cup B)$ by $f(n) = \begin{cases} a_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ b_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$

Clearly from the diagram above, f is one to one and onto. Hence, $(A \cup B)$ is denumerable. ■

6. The set of rational numbers is denumerable (countable).

Proof. $\mathbf{Q}^+ \cup \{0\}$ is the union of a denumerable set and a finite set, hence it is denumerable.

The entire set of rationals can be expressed as $\mathbf{Q} = (\mathbf{Q}^+ \cup \{0\}) \cup \mathbf{Q}^-$, which is the union of two denumerable sets, hence denumerable. ■

7. The set of real numbers in the interval $[0,1]$ is uncountable (non-denumerable).

Proof. (By contradiction) Suppose, for the sake of deriving a contradiction, that the set of real numbers in the interval $[0,1]$ is denumerable.

Then the entire set of real numbers in the interval $[0,1]$ can be ordered (i.e., put into a one to one correspondence with the natural numbers).

Consider such an exhaustive ordering (or list):

$x_1 = 0.x_{11}x_{12}x_{13}\ldots$

$x_2 = 0.x_{21}x_{22}x_{23}\ldots$

$x_3 = 0.x_{31}x_{32}x_{33}\ldots$

$\vdots$

$x_n = 0.x_{n1}x_{n2}x_{n3}\ldots x_{nn}\ldots$

$\vdots$

Here, x_{ij} is the jth decimal digit of x_i.

Also, if x_i can be written in terminating and non-terminating form (e.g., 0.5 can be written as $0.499999\ldots$), then we choose the non-terminating form.

(Note that if we follow this convention, that $0 = 0.00000\ldots$ and $1 = 0.99999\ldots$)

Define $y \in [0,1]$ as follows:

$y = 0.y_1y_2y_3\ldots y_n\ldots$ where y_i is the i^{th} decimal digit of y and y_i is given by

$$y_i = \begin{cases} 5 & \text{if } x_{ii} \neq 5 \\ 6 & \text{if } x_{ii} = 5 \end{cases}$$

Again, note that $0 < y < 1$.

Our hypothesis has led us to conclude that the list above is an *exhaustive* list of real numbers in the interval $[0,1]$. Hence, y must be on the list somewhere.

Observe, however, that for $j = 1,2,3,4,\ldots \ \ y \neq x_j$

(If y *were* equal to x_j for some $j = 1,2,3,4,\ldots$, then they would necessarily have the same digit at the j^{th} decimal place. However, we have defined y in such a way that this can't happen.)

Hence, y is not on the list above.

This contradicts our assumption that the entire set of real numbers in the interval $[0,1]$ is denumerable and can, therefore, be placed in an ordered list.

Since the assumption that the entire set of real numbers in the interval $[0, 1]$ is denumerable leads to a contradiction, the assumption must be false.

Hence, the entire set of real numbers in the interval $[0, 1]$ is non-denumerable (uncountable). ■