Proofs Involving Sets #2 (Equality) - Solutions

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Instructions. Prove the following.

1. $U \cap A = A$

   Proof. We must show that:

   (a) $U \cap A \subseteq A$

   and

   (b) $A \subseteq U \cap A$

   a. $\boxed{U \cap A \subseteq A}$

   This is always true — $(A \cap \textit{ anything }) \subseteq A$

   b. $\boxed{A \subseteq U \cap A}$

   Let $x \in A$

   By definition of Universal set, $x \in U$

   $\Rightarrow x \in U$ and $x \in A$

   $\Rightarrow x \in U \cap A$

   i.e., $x \in A \Rightarrow x \in U \cap A$

   Hence, $A \subseteq U \cap A$ ■

2. $(A \cap B)^c = A^c \cup B^c$

Proof. We must show that:

(a) $(A \cap B)^c \subseteq A^c \cup B^c$

and

(b) $A^c \cup B^c \subseteq (A \cap B)^c$

a. $\boxed{(A \cap B)^c \subseteq A^c \cup B^c}$

Let $x \in (A \cap B)^c$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A^c$ or $x \in B^c$

$\Rightarrow x \in A^c \cup B^c$

We have shown that $x \in (A \cap B)^c \Rightarrow x \in A^c \cup B^c$

Therefore, $(A \cap B)^c \subseteq A^c \cup B^c$

b. $\boxed{A^c \cup B^c \subseteq (A \cap B)^c}$

Let $x \in A^c \cup B^c$

$\Rightarrow x \in A^c$ or $x \in B^c$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \in (A \cap B)^c$

We have shown that $x \in A^c \cup B^c \Rightarrow x \in (A \cap B)^c$

Therefore, $A^c \cup B^c \subseteq (A \cap B)^c$ ■

3. $A \subseteq B \Rightarrow (A \cap B) = A$

Let the hypothesis be given. (i.e., let $A \subseteq B$)

We must show that:

a. $(A \cap B) \subseteq A$

and

b. $A \subseteq (A \cap B)$

a. $\boxed{(A \cap B) \subseteq A}$

This is *always* true — $(A \cap \textit{anything}\ ) \subseteq A$

b. $\boxed{A \subseteq (A \cap B)}$

Let $x \in A$.

$\Rightarrow x \in B$ (Because $A \subseteq B$, by hypothesis).

$\Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in A \cap B$

We have shown that $x \in A \Rightarrow x \in A \cap B$

Therefore, $A \subseteq (A \cap B)$ ■

4. $A \subseteq B \Rightarrow (A \cup B) = B$

Proof. Let the hypothesis be given. (i.e., let $A \subseteq B$)

We must show that:

a. $(A \cup B) \subseteq B$

and

b. $B \subseteq (A \cup B)$ (This is *always* true.)

a. $\boxed{(A \cup B) \subseteq B}$

Let $x \in (A \cup B)$.

$\Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in B$ (because $A \subseteq B$ by hypothesis.)

i.e., $x \in B$

We have shown that $x \in (A \cup B) \Rightarrow x \in B$.

Therefore, $(A \cup B) \subseteq B$

b. $\boxed{B \subseteq (A \cup B)}$

This is *always* true — $B \subseteq (B \cup \textit{anything})$ ■

5. $A^c \setminus B^c = B \setminus A$

Proof. We must show that:

a. $A^c \setminus B^c \subseteq B \setminus A$

and

b. $B \setminus A \subseteq A^c \setminus B^c$ (This is *always* true.)

a. $\boxed{A^c \setminus B^c \subseteq B \setminus A}$

Let $x \in A^c \setminus B^c$

$\Rightarrow x \in A^c$ and $x \notin B^c$

$\Rightarrow x \notin A$ and $x \in B$

i.e, $x \in B$ and $x \notin A$

$\Rightarrow x \in B \setminus A$

b. $\boxed{B \setminus A \subseteq A^c \setminus B^c}$

Let $x \in B \setminus A$

$\Rightarrow x \in B$ and $x \notin A$

$\Rightarrow x \notin B^c$ and $x \in A^c$

i.e, $x \in A^c$ and $x \notin B^c$

$\Rightarrow x \in A^c \setminus B^c$ ■