MTH 1125 (2 pm - Pod B) Test #3

FALL 2020

Pat Rossi Name ______________________________

Instructions. Show CLEARLY how you arrive at your answers.

1. $f(x) = x^3 + 3x^2 + 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums. (Caution - there are **two** critical numbers. Make sure you get them both!)

   i. Compute $f'(x)$ and find the critical numbers

   $f'(x) = 3x^2 + 6x$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 0$ and solve for x

   $\Rightarrow f'(x) = 3x^2 + 6x = 0$

   $\Rightarrow 3x(x+2) = 0$

   $\Rightarrow 3x = 0 \quad \text{or} \quad (x+2) = 0$

   $\Rightarrow x = 0$ and $x = -2$ are critical numbers.

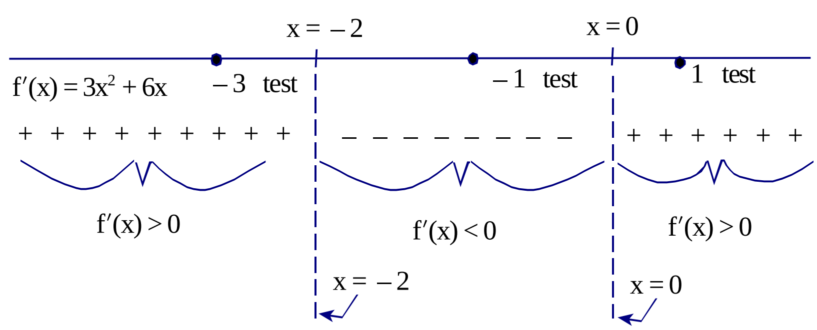
   "Type b" $(f'(c)$ is undefined)

   Look for x-value that causes division by zero.

   No "type b" critical numbers

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

3. Pick a "test point" from each interval to plug into $f'(x)$



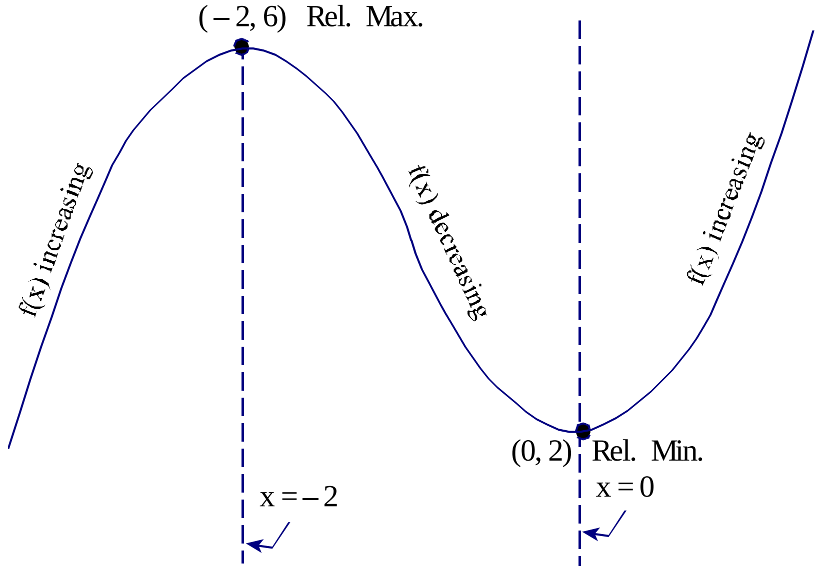
$f(x)$ is **increasing** on the interval(s) $(-\infty, -2)$ and $(0, \infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-2, 0)$

(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(-2, f(-2)) = (-2, 6)$
Rel Min $(0, f(0)) = (0, 2)$

2. $f(x) = x^4 + 8x^3 - 30x^2 + 6x + 3$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection.

   1. Compute $f''(x)$ and find possible points of inflection.

      $f'(x) = 4x^3 + 24x^2 - 60x + 6$

      $f''(x) = 12x^2 + 48x - 60$

      Find possible points of inflection:

      a. "Type a" $(f''(x) = 0)$

         Set $f''(x) = 0$

         $\Rightarrow f''(x) = 12x^2 + 48x - 60 = 0$

         $\Rightarrow x^2 + 4x - 5 = 0$

         $\Rightarrow (x+5)(x-1) = 0$

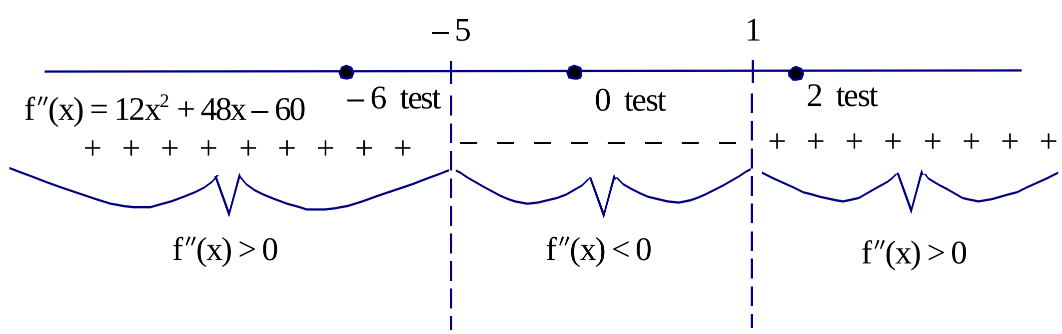
         $x = -5, 1$ possible "type a" points of inflection

      b. "Type b" $(f''(x)$ undefined)

         No "Type b" points of inflection

   2. Draw a "sign graph" of $f''(x)$, using the possible points of inflection to partition the x-axis.

   3. Select a test point from each interval and plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-\infty, -5)$ and $(1, \infty)$
(because $f''(x)$ is positive on these intervals)

$f(x)$ is **concave down** on the interval $(-5, 1)$
(because $f''(x)$ is negative on this interval)

Since $f(x)$ changes concavity at $x = -5$ and $x = 1$, the points:
$(-5, f(-5)) = (-5, -1152)$
and
$(1, f(1)) = (1, -12)$ **are** points of inflection.

3. $f(x) = 2x^3 + 15x^2 - 84x + 3$ on the interval $[-2, 3]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

Note: [1] $f(x)$ is continuous (since it is a polynomial) on the [2]closed, [3]finite interval $[-2, 3]$. Therefore, we can use the Absolute Max/Min Value Test.

i. Compute $f'(x)$ and find the critical numbers.

$f'(x) = 6x^2 + 30x - 84$

a. "Type a" ($f'(x) = 0$)

$f'(x) = 6x^2 + 30x - 84 = 0$

$\Rightarrow x^2 + 5x - 14 = 0$

$\Rightarrow (x + 7)(x - 2) = 0$

$\Rightarrow x = -7, 2$ are "type a" critical numbers

Since $x = -7$ is not in the interval $[-2, 3]$, we discard it as a critical number.

b. "Type b" ($f'(x)$ is undefined)

No "Type b" critical numbers

ii. Plug endpoints and critical numbers into $f(x)$ (the *original* function)

$f(-2) = 2(-2)^3 + 15(-2)^2 - 84(-2) + 3 = 215$ ← Abs Max Value

$f(2) = 2(2)^3 + 15(2)^2 - 84(2) + 3 = -89$ ← Abs Min Value

$f(3) = 2(3)^3 + 15(3)^2 - 84(3) + 3 = -60$

The Abs Max Value is 215 (attained at $x = -2$)
The Abs Min Value is -89 (attained at $x = 2$)

4. $f(x) = \frac{1}{7}x^{\frac{14}{5}} - 2x^{\frac{4}{5}} + 1$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

1. Compute $f'(x)$ and find the critical numbers

$$f'(x) = \frac{2}{5}x^{\frac{9}{5}} - \frac{8}{5}x^{-\frac{1}{5}} = \frac{2x^{\frac{9}{5}}}{5} - \frac{8}{5x^{\frac{1}{5}}} = \frac{2x^{\frac{9}{5}}}{5}\frac{x^{\frac{1}{5}}}{x^{\frac{1}{5}}} - \frac{8}{5x^{\frac{1}{5}}} = \frac{2x^2-8}{5x^{\frac{1}{5}}}$$

i.e., $f'(x) = \frac{2x^2-8}{5x^{\frac{1}{5}}}$

a. "Type a" $(f'(c) = 0)$

Set $f'(x) = 0$ and solve for x

$\Rightarrow f'(x) = \frac{2x^2-8}{5x^{\frac{1}{5}}} = 0$

$\Rightarrow 2x^2 - 8 = 0$

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x+2)(x-2) = 0$

$\Rightarrow x = -2$ and $x = 2$ are critical numbers.

b. "Type b" ($f'(c)$ is undefined)

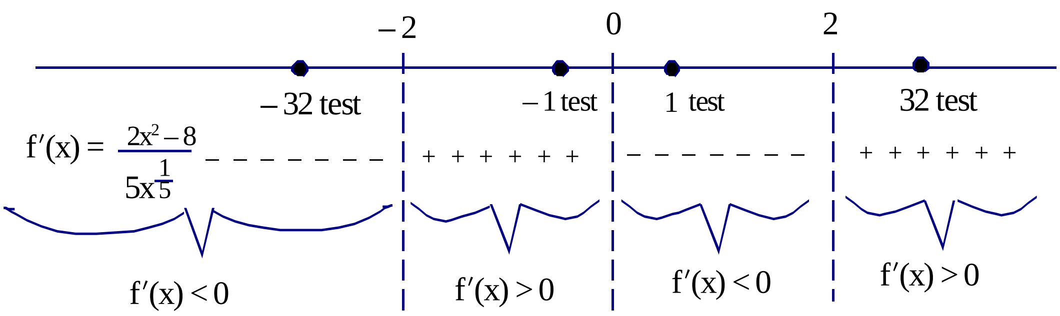
Look for x-value that causes division by zero.

$\Rightarrow 5x^{\frac{1}{5}} = 0$

$\Rightarrow x = 0$ "type b" critical number

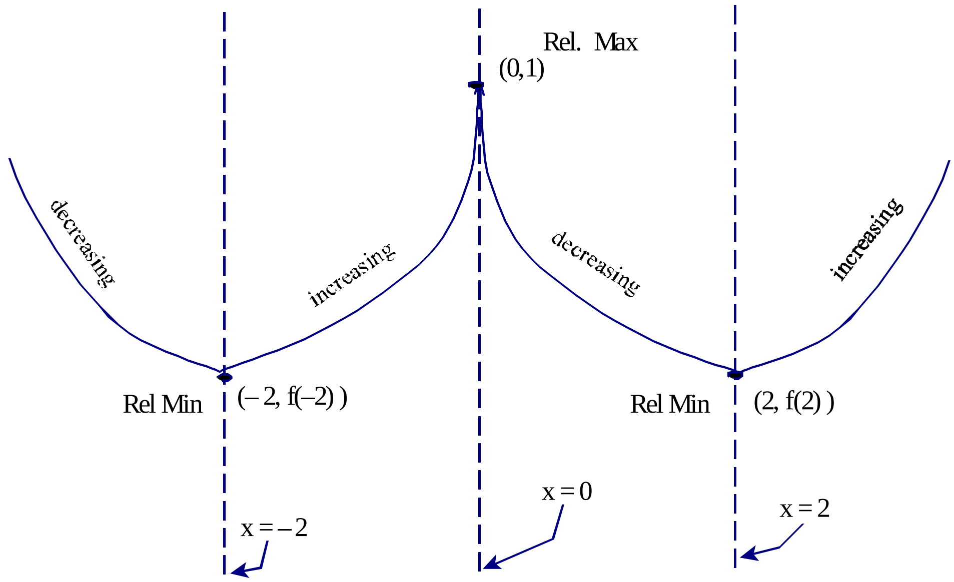
2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

3. Pick a "test point" from each interval to plug into $f'(x)$



> $f(x)$ is **increasing** on the interval(s) $(-2, 0)$ and $(2, \infty)$
>
> (because $f'(x)$ is positive on these intervals)
>
> $f(x)$ is **decreasing** on the interval(s) $(-\infty, -2)$ and $(0, 2)$
>
> (because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



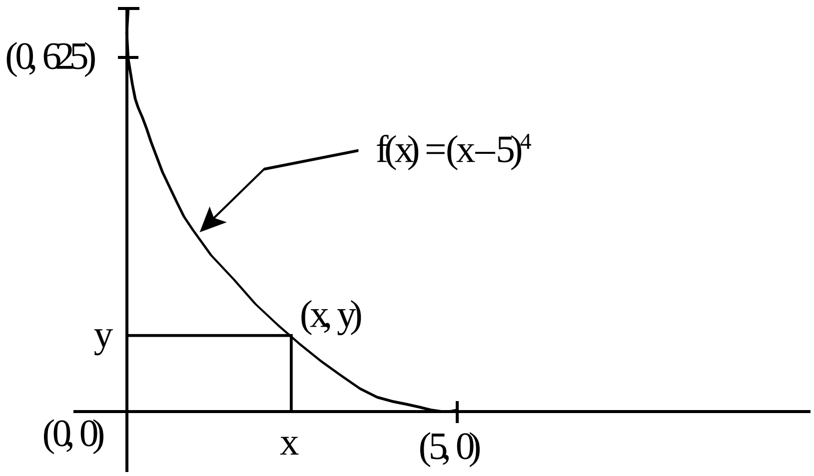
Rel Minimums: $(-2, f(-2)) = \left(-2, \frac{1}{7}(-2)^{\frac{14}{5}} - 2(-2)^{\frac{4}{5}} + 1\right)$

and $(2, f(2)) = \left(2, \frac{1}{7}(2)^{\frac{14}{5}} - 2(2)^{\frac{4}{5}} + 1\right)$

Rel Maximum: $(0, f(0)) = (0, 1)$

5. A rectangle is inscribed in the region bounded by the positive x-axis, the positive y-axis, and the graph of $f(x) = (x-5)^4$ as shown below. Determine the value of x that makes the area of the rectangle as large as possible.

When you get the Area function $A(x)$, **do not simplify** before computing $A'(x)$. When finding the critical numbers, **compute** $A'(x)$ **without simplifying** $A(x)$



1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = xy$

   a. Draw a picture where relevant.

      (Done)

2. Express A as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that the point (x, y) must be on the graph of $f(x) = (x-5)^4$.

   Hence, the y-coordinate of the point (x, y) is $y = (x-5)^4$.

   Plug this into the equation $A = xy$

   $\Rightarrow A(x) = x(x-5)^4$

3. Determine the restrictions on the independent variable x.

   From the picture, $0 \leq x \leq 5$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0, 5]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = \frac{d}{dx}\left[x(x-5)^4\right] = \underbrace{(x-5)^4 + 4x(x-5)^3}_{\text{Product Rule}}$

      i.e., $A'(x) = (x-5)^4 + 4x(x-5)^3$

      a. "Type a" $(f'(c) = 0)$

         $\Rightarrow A'(x) = (x-5)^4 + 4x(x-5)^3 = 0$

         $\Rightarrow (x-5)^4 + 4x(x-5)^3 = 0$

         $\Rightarrow (x-5)^3[(x-5) + 4x] = 0$

         $\Rightarrow (x-5)^3(5x-5) = 0$

         $\Rightarrow x = 5$ and $x = 1$ are critical numbers

      b. "Type b" ($f'(c)$ is undefined)

         Look for x-values that cause division by zero in $f'(x)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0) = (0)((0)-5)^4 = 0$

      $A(1) = (1)((1)-5)^4 = 256 \longleftarrow$ Abs Max Value

      $A(5) = (5)((5)-5)^4 = 0$

5. Make sure that we've answered the original question.

   1. "Determine the value of x ..."

      $x = 1$