MTH 3318 Test #1 - Solutions

FALL 2024

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Instructions. Fully document your work.

For problems 1 - 2 prove one using Mathematical Induction.

1. $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

   i.e. $\sum_{i=1}^{n} (2i - 1) = n^2$ (This is $P(n)$)

   Proof.

   Step #1: Show that $P(n)$ is true for $n = 1$

   $\sum_{i=1}^{1} (2i - 1) = (2(1) - 1) = 1 = (1)^2$ True.

   Step #2: Assume that $P(n)$ is true for $n = k$, and show that $P(n)$ is true for $n = k+1$

   i.e., Assume that $\sum_{i=1}^{k} (2i - 1) = k^2$ for some natural number k, and show that $\sum_{i=1}^{k+1} (2i - 1) = (k+1)^2$

   Observe:

   $$\sum_{i=1}^{k+1} (2i - 1) = \underbrace{\sum_{i=1}^{k} (2i - 1) + (2(k+1) - 1) = k^2 + (2(k+1) - 1)}_{\text{by Induction Hypothesis}}$$

   $$= k^2 + 2k + 1 = (k+1)^2$$

   i.e., $\sum_{i=1}^{k+1} (2i - 1) = (k+1)^2$

   Hence, $\sum_{i=1}^{n} (2i - 1) = n^2$ for all natural numbers, n. ■

2. $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

i.e. $\sum_{j=1}^{n}\frac{1}{(2j-1)(2j+1)}=\frac{n}{2n+1}$ (This is $P(n)$)

Proof.

Step #1: Show that $P(n)$ is true for $n=1$

$\sum_{i=1}^{1}\frac{1}{(2i-1)(2i+1)}=\frac{1}{(2(1)-1)(2(1)+1)}=\frac{1}{3}=\frac{(1)}{2(1)+1}$ True.

Step #2: Assume that $P(n)$ is true for $n=k$, and show that $P(n)$ is true for $n=k+1$

i.e., Assume that $\sum_{i=1}^{k}\frac{1}{(2i-1)(2i+1)}=\frac{k}{2k+1}$ for some natural number k, and show that $\sum_{i=1}^{k+1}\frac{1}{(2i-1)(2i+1)}=\frac{k+1}{2(k+1)+1}$

i.e., $\sum_{i=1}^{k+1}\frac{1}{(2i-1)(2i+1)}=\frac{k+1}{2k+3}$

Observe:

$$\sum_{i=1}^{k+1}\frac{1}{(2i-1)(2i+1)}=\sum_{i=1}^{k}\frac{1}{(2i-1)(2i+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \text{ (by Induction Hypothesis)}$$

$$=\frac{k}{2k+1}\cdot\frac{2k+3}{2k+3}+\frac{1}{(2k+1)(2k+3)}=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}=\frac{(k+1)}{(2k+3)}$$

i.e., $\sum_{i=1}^{k+1}\frac{1}{(2i-1)(2i+1)}=\frac{k+1}{2k+3}$

Hence, $\sum_{i=1}^{n}\frac{1}{(2i-1)(2i+1)}=\frac{n}{2n+1}$ for all natural numbers, n. ■

For problems 3 - 5 prove one using Mathematical Induction.

3. $1+5+9+\ldots+(4n-3)=2n^2-n$

   i.e., $\sum_{i=1}^{n}(4i-3)=2n^2-n$ (This is $P(n)$)

   Proof.

   Step #1: Show that $P(n)$ is true for $n=1$

   $\sum_{i=1}^{1}(4i-3)=4(1)-3=1=2(1)^2-(1)$ True.

   Step #2: Assume that $P(n)$ is true for $n=k$, and show that $P(n)$ is true for $n=k+1$

   i.e., Assume that $\sum_{i=1}^{k}(4i-3)=2k^2-k$ for some natural number k, and show that $\sum_{i=1}^{k+1}(4i-3)=2(k+1)^2-(k+1)$

   Equivalently, show that $\sum_{i=1}^{k+1}(4i-3)=2k^2+3k+1$

   Observe:

   $$\sum_{i=1}^{k+1}(4i-3)=\underbrace{\sum_{i=1}^{k}(4i-3)+4[(k+1)-3]=\left(2k^2-k\right)+4[(k+1)-3]}_{\text{by Induction Hypothesis}}$$

   $$=(2k^2-k)+4k+4-3$$

   $$=2k^2+3k+1$$

   i.e., $\sum_{i=1}^{k+1}(4i-3)=2k^2+3k+1$

   Hence, $\sum_{i=1}^{n}(4i-3)=2n^2-n$ for all natural numbers, n. ■

4. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

i.e. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ (This is $P(n)$)

Proof.

Step #1: Show that $P(n)$ is true for $n = 1$.

$\sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{(1)[(1)+1][2(1)+1]}{6}$. True.

Step #2: Assume that $P(n)$ is true for $n = k$, and show that $P(n)$ is true for $n = k + 1$.

i.e., Assume that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ and show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6}$.

i.e., show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Observe:

$$\sum_{i=1}^{k+1} i^2 = \underbrace{\sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2}_{\text{by Induction Hypothesis}} = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{k(k+1)(2k+1)+6(k+1)^2}{6} = \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)\left[2k^2+7k+6\right]}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

i.e., $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Hence, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$; $\forall n \in \mathbf{N}$ ■

5. $\frac{n^4}{4} < 1^3 + 2^3 + 3^3 + \ldots + n^3$ all natural numbers, n. (This is $P(n)$)

Proof.

Step #1: Show that $P(n)$ is true for $n = 1$.

$\frac{(1)^4}{4} = \frac{1}{4} < 1^3$

i.e., $\frac{(1)^4}{4} < 1^3$. True.

Step #2: Assume that $P(n)$ is true for $n = k$, and show that $P(n)$ is true for $n = k + 1$.

i.e., Assume that $\frac{k^4}{4} < 1^3 + 2^3 + 3^3 + \ldots + k^3$

and show that $\frac{(k+1)^4}{4} < 1^3 + 2^3 + 3^3 + \ldots + (k+1)^3$

Remark: Our argument may be easier to follow if we "swap the sides" of the inequality. (i.e., if we show that: $1^3 + 2^3 + 3^3 + \ldots + (k+1)^3 > \frac{(k+1)^4}{4}$)

Observe: $\underbrace{1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 > \frac{k^4}{4} + (k+1)^3}_{\text{By our induction hypothesis}} = \frac{k^4}{4} + (k^3 + 3k^2 + 3k + 1)$

$$= \frac{k^4}{4} + \frac{4\left(k^3+3k^2+3k+1\right)}{4} = \frac{k^4+4k^3+12k^2+12k+4}{4} > \frac{k^4+4k^3+6k^2+4k+1}{4} = \frac{(k+1)^4}{4}$$

i.e., $1^3 + 2^3 + 3^3 + \ldots + (k+1)^3 > \frac{(k+1)^4}{4}$

For problems 6 - 7, prove one using Mathematical Induction:

6. $n(n+1)$ is divisible by 2 for all natural numbers, n. (This is $P(n)$)

Proof.

First, note that a natural number n is divisible by 2 if there exists a natural number m such that $n = 2m$

Step #1: Show that $P(n)$ true for $n = 1$.

$1((1)+1) = 2 = 2 \cdot 1$

Thus, $n(n+1)$ is divisible by 2, for $n = 1$.

i.e., $1((1)+1) = 2 = 2 \cdot 1$ True.

Step #2: Assume that $P(n)$ is true for $n = k$, and show that $P(n)$ is true for $n = k+1$

i.e., Assume that $k(k+1)$ is divisible by 2, and show that

$(k+1)[(k+1)+1]$ is divisible by 2.

i.e., Assume that $k(k+1) = 2m$, and show that

$(k+1)(k+2)$ is divisible by 2.

Observe: $(k+1)(k+2) = (k+1)k + (k+1)2 = \underbrace{k(k+1) + 2(k+1) = 2m+2}_{\text{by Ind. Hyp.}} =$

$2(m+1)$.

i.e., $(k+1)(k+2) = 2(m+1)$.

i.e., $(k+1)(k+2)$ is divisible by 2.

Hence, $n(n+1)$ is divisible by 2 for all natural numbers, n. ■

7. Given that $\frac{d}{dx}\left[x^0\right] = 0$ and $\frac{d}{dx}\left[x^1\right] = 1$, show that $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$. (This is $P(n)$).

You may use the product rule: $\frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)\,g(x) + g'(x)\,f(x)$.

Proof.

Step #1: Show that $P(n)$ is true for $n = 1$.

$\frac{d}{dx}\left[x^1\right] = 1 = x^0 = x^{1-1}$ True.

Step #2: Assume that $P(n)$ is true for $n = k$, and show that $P(n)$ is true for $n = k+1$

i.e., Assume that $\frac{d}{dx}\left[x^k\right] = kx^{k-1}$ and show that $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^{(k+1)-1}$

i.e., show that $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^k$

Observe:

$$\frac{d}{dx}\left[x^{k+1}\right] = \frac{d}{dx}\left[x^k \cdot x\right] = \underbrace{\frac{d}{dx}\left[x^k\right] \cdot x + \frac{d}{dx}\left[x\right] \cdot x^k}_{\text{product rule}} = \underbrace{kx^{k-1}}_{\text{Ind Hyp}} \cdot x + \underbrace{1}_{\text{given}} \cdot x^k$$

$$= kx^k + x^k = (k+1)\,x^k$$

i.e. $\frac{d}{dx}\left[x^{k+1}\right] = (k+1)\,x^k$

Hence, $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$ for all natural numbers n. ■

For problems 8 - 9, prove one using Mathematical Induction:

8. $(1+x)^n \geq 1+nx$ for any natural number n and any real number $x \geq -1$.
(This is $P(n)$)

Proof.

Step #1: Show that $P(n)$ is true for $n=1$

$(1+x)^1 = 1+x \geq 1+(1)x$ True.

Step #2: Assume $P(n)$ is true for $n=k$, and show that $P(n)$ is true for $n=k+1$

i.e., Assume that $(1+x)^k \geq 1+kx$ for some natural number k, and show that

$(1+x)^{k+1} \geq 1+(k+1)x$

Observe:

$$(1+x)^{k+1} = \underbrace{(1+x)^k(1+x) \geq (1+kx)(1+x)}_{\text{by Induction Hypothesis}} = 1+kx+x+kx^2$$

$$= 1+(k+1)x + \underbrace{kx^2}_{kx^2\geq 0} \geq 1+(k+1)x$$

i.e., $(1+x)^{k+1} \geq 1+(k+1)x$

Hence, $(1+x)^n \geq 1+nx$ for all natural numbers n and any real number $x \geq -1$ ■

Remark: Our proof hinged on two subtle points:

First, since k is a natural number (hence greater than zero) and $x^2 \geq 0$ for ALL real numbers x, it follows that $kx^2 \geq 0$.

Second, since it is given that $x \geq -1$ (or equivalently, $(1+x) \geq 0$), the direction of the inequality, $(1+x)^k \geq 1+kx$, is preserved when both sides are multiplied by $(1+x)$ during the application of the induction hypothesis.

9. Given that $|x_1 + x_2| \le |x_1| + |x_2|$ (the Triangle Inequality); Prove by induction that: $|x_1 + x_2 + x_3 + \ldots + x_n| \le |x_1| + |x_2| + |x_3| + \ldots + |x_n|$ (the General Triangle Inequality). (This is $P(n)$)

Proof.

Step #1: Show that $P(n)$ is true for $n = 1$.

$|x_1| \le |x_1|$. True.

Step #2: Assume that $P(n)$ is true for $n = k$, and show that $P(n)$ is true for $n = k + 1$.

i.e., Assume that $|x_1 + x_2 + x_3 + \ldots + x_k| \le |x_1| + |x_2| + |x_3| + \ldots + |x_k|$ and show that $|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \le |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

Observe: $\underbrace{|(x_1 + x_2 + x_3 + \ldots + x_k) + x_{k+1}| \le |x_1 + x_2 + x_3 + \ldots + x_k| + |x_{k+1}|}_{\text{from Triangle Inequality}}$

$\le \underbrace{|x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|}_{\text{by Ind. Hyp.}}$.

i.e., $|x_1 + x_2 + x_3 + \ldots + x_k + x_{k+1}| \le |x_1| + |x_2| + |x_3| + \ldots + |x_k| + |x_{k+1}|$.

Hence, $|x_1 + x_2 + x_3 + \ldots + x_n| \le |x_1| + |x_2| + |x_3| + \ldots + |x_n|$ for all natural numbers, n. ■