MTH 1126 Test #3 - Solutions

Spring 2023

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Show CLEARLY how you arrive at your answers.

1. $\int \frac{5x+31}{x^2-3x-10}dx =$

Note that $\int \frac{5x+31}{x^2-3x-10}dx$ does not fit the form: $\int \frac{1}{u}du$

Therefore, we decompose $\frac{5x+31}{x^2-3x-10}$ into the sum of simpler quotients:

1. Make sure that deg(numerator) $\leq$ deg(denominator)

2. Factor the denominator.

$\frac{5x+31}{x^2-3x-10} = \frac{5x+31}{(x+2)(x-5)}$

3. For each linear factor $(x+c)$, form the term $\frac{C_1}{x+c}$

$\frac{5x+31}{x^2-3x-10} = \frac{5x+31}{(x+2)(x-5)} = \frac{C_1}{x+2} + \frac{C_2}{x-5}$

4. Solve for the constants

$\frac{5x+31}{(x+2)(x-5)} = \frac{C_1}{x+2} + \frac{C_2}{x-5}$

$\Rightarrow \frac{5x+31}{(x+2)(x-5)}(x+2)(x-5) = \frac{C_1}{x+2}(x+2)(x-5) + \frac{C_2}{x-5}(x+2)(x-5)$

i.e., $5x+31 = C_1(x-5) + C_2(x+2)$

Plug in "strategic values" of x to find the values of the constants.

$\boxed{x=-2}$

$\Rightarrow 21 = -7C_1$

$\Rightarrow \boxed{C_1 = -3}$

$\boxed{x=5}$

$\Rightarrow 56 = 7C_2$

$\Rightarrow \boxed{C_2 = 8}$

Thus, $\frac{5x+31}{x^2-3x-10} = \frac{C_1}{x+2} + \frac{C_2}{x-5} = -\frac{3}{x+2} + \frac{8}{x-5}$

i.e., $\frac{5x+31}{x^2-3x-10} = -\frac{3}{x+2} + \frac{8}{x-5}$

Consequently, $\int \frac{5x+31}{x^2-3x-10}dx = \int \left(-\frac{3}{x+2} + \frac{8}{x-5}\right) dx = -3\int \frac{1}{x+2}dx + 8\int \frac{1}{x-5}dx$

$$= -3\ln|x+2| + 8\ln|x-5| + C$$

$$\boxed{\int \frac{5x+31}{x^2-3x-10}dx = -3\ln|x+2| + 8\ln|x-5| + C}$$

2. $\lim_{x\to\infty} \frac{\ln(x)}{x^2} =$

   $\lim_{x\to\infty} \frac{\ln(x)}{x^2} \sim \frac{\infty}{\infty}$ (Use L'Hopital's Rule)

   $\lim_{x\to\infty} \frac{\ln(x)}{x^2} = \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[x^2]} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$

   $$\boxed{\lim_{x\to\infty} \frac{\ln(x)}{x^2} = 0}$$

3. $\int \frac{3x^2+2x+12}{x^3+4x}dx =$

Note that $\int \frac{3x^2+2x+12}{x^3+4x}dx$ does not fit the form: $\int \frac{1}{u}du$

Therefore, we decompose $\frac{3x^2+2x+12}{x^3+4x}$ into the sum of simpler quotients:

1. Make sure that deg(numerator) $\leq$ deg(denominator)

2. Factor the denominator.

$\frac{3x^2+2x+12}{x^3+4x} = \frac{3x^2+2x+12}{x(x^2+4)}$ (x^2+4 is an "irreducible quadratic")

To see this, note that $x^2 \geq 0$. Therefore $x^2+4 \geq 4$. And consequently, $x^2+4 \neq 0$ for any value of x. This means that x^2+4 cannot be factored.

Alternatively, note that if we plug the coefficients of x^2+4 into the quadratic formula, we get: $x = \frac{-0 \pm \sqrt{0^2-4(1)(4)}}{2(1)} = \frac{-0 \pm \sqrt{-16}}{2}$.

The fact that we get a negative under the radical tells us that x^2+4 is irreducible.

3. For each linear factor $(x+c)$, form the term $\frac{C_1}{x+c}$.

3.a. For each irreducible quadratic ax^2+bx+c, form the term $\frac{Ax+B}{ax^2+bx+c}$

$\frac{3x^2+2x+12}{x^3+4x} = \frac{3x^2+2x+12}{x(x^2+4)} = \frac{C}{x} + \frac{Ax+B}{x^2+4}$

4. Solve for the constants

$\frac{3x^2+2x+12}{x(x^2+4)} = \frac{C}{x} + \frac{Ax+B}{x^2+4}$

$\Rightarrow \frac{3x^2+2x+12}{x(x^2+4)}x(x^2+4) = \frac{C}{x}x(x^2+4) + \frac{Ax+B}{x^2+4}x(x^2+4) = C(x^2+4) + (Ax+B)x$

i.e., $3x^2+2x+12 = C(x^2+4) + (Ax+B)x$

Plug in "strategic values" of x to find the values of the constants.

$\boxed{x=0}$

$\Rightarrow 12 = 4C$

$\Rightarrow \boxed{C=3}$

This yields: $3x^2+2x+12 = 3(x^2+4) + (Ax+B)x$

Simplifying both sides we have:

$3x^2+2x+12 = (A+3)x^2 + Bx + 12$

Comparing coefficients of x^2 on both sides, we see that:

$3 = A + 3$

$\Rightarrow \boxed{A = 0}$

Comparing coefficients of x on both sides, we see that:

$\boxed{B = 2}$

Thus, $\frac{3x^2+2x+12}{x(x^2+4)} = \frac{3}{x} + \frac{0x+2}{x^2+4} = \frac{3}{x} + \frac{2}{x^2+4}$

i.e., $\frac{3x^2+2x+12}{x^3+4x} = \frac{3x^2+2x+12}{x(x^2+4)} = \frac{3}{x} + \frac{2}{x^2+4}$

Consequently, $\int \frac{3x^2+2x+12}{x^3+4x}dx = \int \left(\frac{3}{x} + \frac{2}{x^2+4}\right) dx = 3\int \frac{1}{x}dx + 2\int \frac{1}{x^2+4}dx$

$$= 3\ln|x| + \arctan\left(\frac{x}{2}\right) + C$$

$$\boxed{\int \frac{3x^2+2x+12}{x^3+4x}dx = 3\ln|x| + \arctan\left(\frac{x}{2}\right) + C}$$

4. $\int \sin^3(x)\cos^3(x)\,dx =$

We have odd powers of $\sin(x)$ and $\cos(x)$.

Therefore, we can use either $\sin(x)$ or $\cos(x)$ as our "future du."

1. Reserve a factor of $\sin(x)$ to serve as our "future du."

$$= \int \sin^2(x)\cos^3(x)\underbrace{\sin(x)\,dx}_{\text{"future du"}}$$

This means that we intend to let $u = \cos(x)$

2. Convert remaining sines into cosines

$$= \int \left(1-\cos^2(x)\right)\cos^3(x)\sin(x)\,dx = \int \left(\cos^3(x)-\cos^5(x)\right)\sin(x)\,dx$$

$$\boxed{\begin{array}{ll} \text{Let} & u = \cos(x) \\ \Rightarrow & \frac{du}{dx} = -\sin(x) \\ \Rightarrow & du = -\sin(x)\,dx \\ \Rightarrow & -du = \sin(x)\,dx \end{array}}$$

$$= \int \underbrace{\left(\cos^3(x)-\cos^5(x)\right)}_{u^3-u^5}\underbrace{\sin(x)\,dx}_{-du} = \int \left(u^3-u^5\right)(-du) = \int \left(u^5-u^3\right)du = \frac{u^6}{6}-\frac{u^4}{4}+C$$

$$= \frac{\cos^6(x)}{6}-\frac{\cos^4(x)}{4}+C$$

$$\boxed{\int \sin^3(x)\cos^3(x)\,dx = \frac{\cos^6(x)}{6}-\frac{\cos^4(x)}{4}+C}$$

Alternative Solution appears on the next page

(Ex 4 continued)

$\int \sin^3(x)\cos^3(x)\,dx =$

We have odd powers of $\sin(x)$ and $\cos(x)$.

Therefore, we can use either $\sin(x)$ or $\cos(x)$ as our "future du."

1. Reserve a factor of $\cos(x)$ to serve as our "future du."

$$= \int \sin^3(x)\cos^2(x)\underbrace{\cos(x)\,dx}_{\text{"future du"}}$$

This means that we intend to let $u = \sin(x)$

2. Convert remaining cosines into sines

$$= \int \sin^3(x)\left(\cos^2(x)\right)\underbrace{\cos(x)\,dx}_{\text{"future du"}}$$

$$= \int \sin^3(x)\left(1-\sin^2(x)\right)\cos(x)\,dx$$

$$= \int \left(\sin^3(x)-\sin^5(x)\right)\cos(x)\,dx$$

Let	$u = \sin(x)$
$\Rightarrow$	$\frac{du}{dx} = \cos(x)$
$\Rightarrow$	$du = \cos(x)\,dx$

$$= \int \left(u^3 - u^5\right)du$$

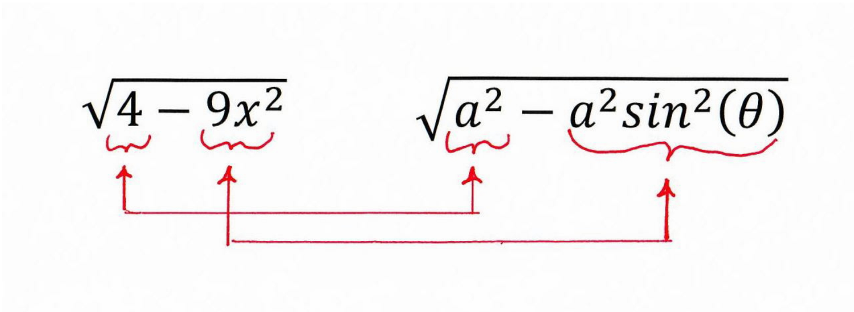
$$= \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$$

$$= \frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) + C$$

$$\boxed{\int \sin^3(x)\cos^3(x)\,dx = \frac{1}{4}\sin^4(x) - \frac{1}{6}\sin^6(x) + C}$$

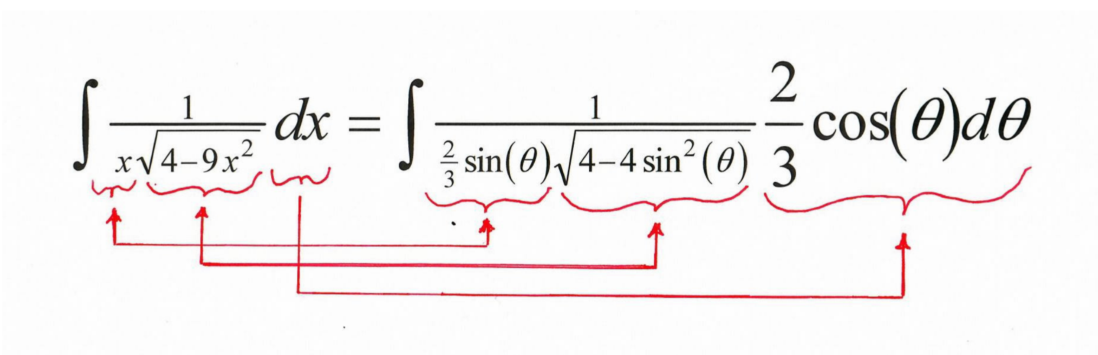
5. $\int \frac{1}{x\sqrt{4-9x^2}} dx =$

We match the radical $\sqrt{4-9x^2}$ with the radical $\sqrt{a^2 - a^2\sin^2(\theta)}$



	a^2	$=$	4
$\Rightarrow$	a	$=$	2
	$9x^2$	$=$	$a^2\sin^2(\theta)$
i.e.	$9x^2$	$=$	$4\sin^2(\theta)$
$\Rightarrow$	$3x$	$=$	$2\sin(\theta)$
$\Rightarrow$	x	$=$	$\frac{2}{3}\sin(\theta)$
$\Rightarrow$	$\frac{dx}{d\theta}$	$=$	$\frac{2}{3}\cos(\theta)$
$\Rightarrow$	dx	$=$	$\frac{2}{3}\cos(\theta)\,d\theta$

Rewrite the integral in terms of θ

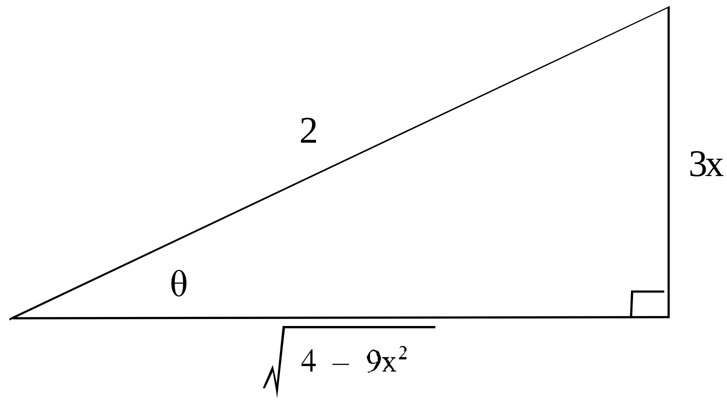


$$\int \frac{1}{x\sqrt{4-9x^2}} dx = \int \frac{1}{\frac{2}{3}\sin(\theta)\sqrt{4-4\sin^2(\theta)}} \frac{2}{3}\cos(\theta)\,d\theta = \int \frac{1}{\frac{2}{3}\sin(\theta)\sqrt{4\cos^2(\theta)}} \frac{2}{3}\cos(\theta)\,d\theta$$

$$= \int \frac{1}{\frac{2}{3}\sin(\theta)2\cos(\theta)} \frac{2}{3}\cos(\theta)\,d\theta = \int \frac{1}{2\sin(\theta)} d\theta = \frac{1}{2}\int \csc(\theta)\,d\theta$$

$$= \frac{1}{2}\ln|\csc(\theta) - \cot(\theta)| + C$$

To convert back to x, recall that $x = \frac{2}{3}\sin(\theta)$ (i.e., $\sin(\theta) = \frac{3x}{2} = \frac{\text{opp}}{\text{hyp}}$)



$$\int \frac{1}{x\sqrt{4-9x^2}}dx = \ldots = \frac{1}{2}\ln\left|\csc(\theta) - \cot(\theta)\right| + C = \frac{1}{2}\ln\left|\frac{2}{3x} - \frac{\sqrt{4-9x^2}}{3x}\right| + C$$

$$\boxed{\int \frac{1}{x\sqrt{4-9x^2}}dx = \frac{1}{2}\ln\left|\frac{2}{3x} - \frac{\sqrt{4-9x^2}}{3x}\right| + C}$$

6. $\lim_{x\to 0^+} x\ln(x) =$

$\lim_{x\to 0^+} x\ln(x)\ \tilde{}\ 0\cdot(-\infty)$

In order to use L'Hôpital's Rule, we must rewrite this expression so that it is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} \sim \frac{-\infty}{\infty}$ We can use L'Hôpital's Rule

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-1}} = \lim_{x\to 0^+} \frac{\frac{d}{dx}[\ln(x)]}{\frac{d}{dx}[x^{-1}]} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{x^{-2}} = \lim_{x\to 0^+} \frac{1}{x}\cdot x^2$$

$$= \lim_{x\to 0^+} x = 0$$

i.e., $\lim_{x\to 0^+} x\ln(x) = 0$

7. $\int x\sin(2x)\,dx =$

Use Integration by Parts

Our "Rules of Thumb" apply

Let u be the portion of the integrand whose derivative is simpler that itself.

$\Rightarrow u = x$

Let dv be the most complicated portion of the integrand that can be integrated

$\Rightarrow dv = \sin(2x)$

	$u = x$		$dv = \sin(2x)\,dx$
$\Rightarrow$	$\frac{du}{dx} = 1$	$\Rightarrow$	$v = \int \sin(2x)\,dx$
$\Rightarrow$	$du = dx$	$\Rightarrow$	$v = -\frac{1}{2}\cos(2x)$

$$\Rightarrow \int x\sin(2x)\,dx = \int u\,dv = uv - \int v\,du = x\left(-\tfrac{1}{2}\cos(2x)\right) - \int\left(-\tfrac{1}{2}\cos(2x)\right)dx$$

$$= -\tfrac{1}{2}x\cos(2x) + \tfrac{1}{2}\int\cos(2x) = -\tfrac{1}{2}x\cos(2x) + \tfrac{1}{2}\left[\tfrac{1}{2}\sin(2x)\right] + C$$

$$= -\tfrac{1}{2}x\cos(2x) + \tfrac{1}{4}\sin(2x) + C$$

$\int x\sin(2x)\,dx = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C$

WOW! Extra (10 pts - all or nothing)

$\int x \arctan(x)\, dx =$

Use Integration by Parts

The "Rule of Thumb" applies here is: Let u be the transcendental function whose derivative is algebraic. (This rule usually "trumps" all others.)

Let dv be everything else

$\Rightarrow u = \arctan(x)$

	$u = \arctan(x)$		$dv = x\,dx$
$\Rightarrow$	$\frac{du}{dx} = \frac{1}{1+x^2}$	$\Rightarrow$	$v = \int x\,dx$
$\Rightarrow$	$du = \frac{1}{1+x^2}dx$	$\Rightarrow$	$v = \frac{1}{2}x^2$

$\int x \arctan(x)\, dx = \int \arctan(x)\, x\,dx = \int u dv = uv - \int v du$

$$= \arctan(x) \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 \frac{1}{1+x^2} dx = \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}\int \frac{x^2}{1+x^2} dx$$

$$= \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}\int \left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right) dx$$

$$= \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}x + \frac{1}{2}\arctan(x) + C$$

$$\int x \arctan(x)\, dx = \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}x + \frac{1}{2}\arctan(x) + C = \frac{1}{2}\left(x^2+1\right)\arctan(x) - \frac{1}{2}x$$