Homework #7 - Direct Products

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Def - Given groups $(G, *_G)$ and $(H, *_H)$, the **Product** of $(G, *_G)$ and $(H, *_H)$, denoted $G \times H$, is the group whose elements are ordered pairs of the form (g, h) such that $g \in G$ and $h \in H$. The product (or sum) of elements in $G \times H$ are computed component-wise, as follows:

$$(g_1, h_1)(g_2, h_2) = (g_1 *_G g_2, h_1 *_H h_2)$$

The Product of three or more groups is defined analogously.

In exercises 1-17, The group $\mathbb{Z}_n$ is the group $(\mathbb{Z}_n, \oplus)$, where $\oplus$ is addition modulo n.

1. List the elements of the group $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

   Note: The elements of $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are ordered triples (a, b, c) such that $a \in \mathbb{Z}_3, b \in \mathbb{Z}_2$, and $c \in \mathbb{Z}_2$.

   Since the second and third components of the group elements (a, b, c) can only be 0 or 1, the easiest way to list the elements is probably to list all group elements having second and third components of 0,0 then 0,1 then 1,0 then 1,1.

   Thus the elements of $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are:

   $\{(0,0,0);(1,0,0);(2,0,0);(0,0,1);(1,0,1);(2,0,1);(0,1,0);(1,1,0);(2,1,0);(0,1,1);(1,1,1);(2,1,1)\}$

2. Determine whether or not $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is cyclic. If it is cyclic, list the generators.

   Recall: $(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k}, \oplus)$, is cyclic exactly when $n_1, n_2, \ldots, n_k$ are "pairwise relatively prime."

   3, 2, 2 are **not** "pairwise relatively prime."

   Therefore, $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is NOT cyclic.

3. Compute the sum of the elements $(2, 1, 0)$ and $(1, 1, 1)$ in the group $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

   Note: The operation on the first component is addition modulo 3

   The operation on the second and third components is addition modulo 2

   $(2, 1, 0) \oplus (1, 1, 1) = ((2 + 1), (1 + 1), (0 + 1)) = (0, 0, 1)$

4. Compute the sum of the elements $(2, 1, 0)$ and $(2, 1, 1)$ in the group $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

   Note: The operation on the first component is addition modulo 3

   The operation on the second and third components is addition modulo 2

   $(2, 1, 0) \oplus (2, 1, 1) = ((2 + 2), (1 + 1), (0 + 1)) = (1, 0, 1)$

5. List the elements of the group $\mathbb{Z}_6 \times \mathbb{Z}_2$

   Note: The elements of $\mathbb{Z}_6 \times \mathbb{Z}_2$ are ordered pairs (a, b) such that $a \in \mathbb{Z}_6$ and $b \in \mathbb{Z}_2$.

   Since the second component of the group elements can only be 0 or 1, the easiest way to list the elements is probably to list all group elements having second component of 0 and then 1.

   Thus the elements of $\mathbb{Z}_6 \times \mathbb{Z}_2$ are:

   $\{(0,0)\,;(1,0)\,;(2,0)\,;(3,0)\,;(4,0)\,;(5,0)\,;(0,1)\,;(1,1)\,;(2,1)\,;(3,1)\,;(4,1)\,;(5,1)\}$

6. Determine whether or not $\mathbb{Z}_6 \times \mathbb{Z}_2$ is cyclic. If it is cyclic, list the generators.

   Recall: $(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k}, \oplus)$, is cyclic exactly when $n_1, n_2, \ldots, n_k$ are "pairwise relatively prime."

   6 and 2 are **not** "pairwise relatively prime."

   Thus, $\mathbb{Z}_6 \times \mathbb{Z}_2$ is NOT cyclic.

7. Compute the sum of the elements $(5, 1)$ and $(4, 0)$ in the group $\mathbb{Z}_6 \times \mathbb{Z}_2$

   Note: The operation on the first component is addition modulo 6

   The operation on the second component is addition modulo 2

   $(5,1) \oplus (4,0) = ((5+4)\,,(1+0)) = (3,1)$

8. Compute the sum of the elements $(3, 1)$ and $(4, 1)$ in the group $\mathbb{Z}_6 \times \mathbb{Z}_2$

   Note: The operation on the first component is addition modulo 6

   The operation on the second component is addition modulo 2

   $(3,1) \oplus (4,1) = ((3+4)\,,(1+1)) = (1,0)$

9. List the elements of the group $\mathbb{Z}_4 \times \mathbb{Z}_3$

   Since the second component of the group elements can only be 0, 1 or 2, the easiest way to list the elements is probably to list all group elements having second component of 0, then 1, and then 2.

   $\{(0,0)\,;(1,0)\,;(2,0)\,;(3,0)\,;(0,1)\,;(1,1)\,;(2,1)\,;(3,1)\,;(0,2)\,;(1,2)\,;(2,2)\,;(3,2)\}$

10. Determine whether or not $\mathbb{Z}_4 \times \mathbb{Z}_3$ is cyclic.

    Recall: $(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k}, \oplus)$, is cyclic exactly when $n_1, n_2, \ldots, n_k$ are "pairwise relatively prime."

    4 and 3 **are** "pairwise relatively prime."

    Therefore, $\mathbb{Z}_4 \times \mathbb{Z}_3$ IS cyclic.

    Recall also: If $(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_k}, \oplus)$ is cyclic, then the generators are of the form:

    $(g_1, g_2, \ldots, g_n)$, where g_i is a generator of $\mathbb{Z}_{n_i}$

    Since the generators of $\mathbb{Z}_4$ are 1 and 3, and the generators of $\mathbb{Z}_3$ are 1 and 2, the generators of $\mathbb{Z}_4 \times \mathbb{Z}_3$ are $(1,1), (1,2), (3,1)$, and $(3,2)$.

11. Compute the sum of the elements $(3,1)$ and $(2,1)$ in the group $\mathbb{Z}_4 \times \mathbb{Z}_3$

    Note: The operation on the first component is addition modulo 4

    The operation on the second component is addition modulo 3

    $(3,1) \oplus (2,1) = ((3+2), (1+1)) = (1,2)$

12. Compute the sum of the elements $(2,2)$ and $(2,2)$ in the group $\mathbb{Z}_4 \times \mathbb{Z}_3$

    Note: The operation on the first component is addition modulo 4

    The operation on the second component is addition modulo 3

    $(2,2) \oplus (2,2) = ((2+2), (2+2)) = (0,1)$

13. Calculate the order of the element $(4,9)$ in the group $\mathbb{Z}_{18} \times \mathbb{Z}_{18}$

    $o(4)$ is the order of 4 as an element of $\mathbb{Z}_{18}$

    $o(4) = \frac{18}{\gcd(4,18)} = \frac{18}{2} = 9$

    $o(9)$ is the order of 9 as an element of $\mathbb{Z}_{18}$

    $o(9) = \frac{18}{\gcd(9,18)} = \frac{18}{9} = 2$

    $o(4,9)$ is the order of $(4,9)$ as an element of $\mathbb{Z}_{18} \times \mathbb{Z}_{18}$

    $o(4,9) = \operatorname{lcm}(o(4), o(9)) = \operatorname{lcm}(9,2) = \frac{9 \cdot 2}{\gcd(9,2)} = \frac{18}{1} = 18$

    $\boxed{o(4,9) = 18}$

    (Note: $\operatorname{lcm}(a,b)$ is the least common multiple of a and b. $\operatorname{lcm}(a,b) = \frac{ab}{\gcd(a,b)}$)

    (Note: $o(m)$ in $\mathbb{Z}_n$ is given by $\frac{n}{\gcd(m,n)}$)

14. Calculate the order of the element $(7,5)$ in the group $\mathbb{Z}_{12} \times \mathbb{Z}_8$

$o(7)$ is the order of 7 as an element of $\mathbb{Z}_{12}$

$o(7) = \frac{12}{\gcd(7,12)} = \frac{12}{1} = 12$

$o(5)$ is the order of 5 as an element of $\mathbb{Z}_8$

$o(5) = \frac{8}{\gcd(5,8)} = \frac{8}{1} = 8$

$o(7,5)$ is the order of $(7,5)$ as an element of $\mathbb{Z}_{12} \times \mathbb{Z}_8$

$o(7,5) = \operatorname{lcm}(o(7), o(5)) = \operatorname{lcm}(12,8) = \frac{12 \cdot 8}{\gcd(12,8)} = \frac{96}{4} = 24$

$\boxed{o(7,5) = 24}$

(Note: $\operatorname{lcm}(a,b)$ is the least common multiple of a and b. $\operatorname{lcm}(a,b) = \frac{ab}{\gcd(a,b)}$)

(Note: $o(m)$ in $\mathbb{Z}_n$ is given by $\frac{n}{\gcd(m,n)}$)

15. Calculate the order of the element $(8,6,4)$ in the group $\mathbb{Z}_{18} \times \mathbb{Z}_9 \times \mathbb{Z}_8$

$o(8)$ is the order of 8 as an element of $\mathbb{Z}_{18}$

$o(8) = \frac{18}{\gcd(8,18)} = \frac{18}{2} = 9$

$o(6)$ is the order of 6 as an element of $\mathbb{Z}_9$

$o(6) = \frac{9}{\gcd(6,9)} = \frac{9}{3} = 3$

$o(4)$ is the order of 4 as an element of $\mathbb{Z}_8$

$o(4) = \frac{8}{\gcd(4,8)} = \frac{8}{4} = 2$

$o(8,6,4)$ is the order of $(8,6,4)$ as an element of $\mathbb{Z}_{18} \times \mathbb{Z}_9 \times \mathbb{Z}_8$

$o(8,6,4) = \operatorname{lcm}(o(8), o(6), o(4)) = \operatorname{lcm}(9,3,2) = \operatorname{lcm}(\operatorname{lcm}(9,3), 2)$

$\operatorname{lcm}(9,3) = \frac{9 \cdot 3}{\gcd(9,3)} = \frac{27}{3} = 9$

$\operatorname{lcm}(\operatorname{lcm}(9,3), 2) = \operatorname{lcm}(9,2) = \frac{9 \cdot 2}{\gcd(9,2)} = \frac{18}{1} = 18$

$\boxed{o(8,6,4) = 18}$

(Note: $\operatorname{lcm}(a,b)$ is the least common multiple of a and b. $\operatorname{lcm}(a,b) = \frac{ab}{\gcd(a,b)}$)

(note also: $\operatorname{lcm}(a,b,c) = \operatorname{lcm}(\operatorname{lcm}(a,b), c)$)

(Note: $o(m)$ in $\mathbb{Z}_n$ is given by $\frac{n}{\gcd(m,n)}$)

16. Calculate the order of the element $(8, 6, 4)$ in the group $\mathbb{Z}_9 \times \mathbb{Z}_{17} \times \mathbb{Z}_{10}$

$o(8)$ is the order of 8 as an element of $\mathbb{Z}_9$

$o(8) = \frac{9}{\gcd(8,9)} = \frac{9}{1} = 9$

$o(6)$ is the order of 6 as an element of $\mathbb{Z}_{17}$

$o(6) = \frac{17}{\gcd(6,17)} = \frac{17}{1} = 17$

$o(4)$ is the order of 4 as an element of $\mathbb{Z}_{10}$

$o(4) = \frac{10}{\gcd(4,10)} = \frac{10}{2} = 5$

$o(8, 6, 4)$ is the order of $(8, 6, 4)$ as an element of $\mathbb{Z}_9 \times \mathbb{Z}_{17} \times \mathbb{Z}_{10}$

$o(8, 6, 4) = \text{lcm}(o(8), o(6), o(4)) = \text{lcm}(9, 17, 5) = \text{lcm}(\text{lcm}(9, 17), 5)$

$\text{lcm}(9, 17) = \frac{9 \cdot 17}{\gcd(9,17)} = \frac{153}{1} = 153$

$\text{lcm}(\text{lcm}(9, 17), 5) = \text{lcm}(153, 5) = \frac{153 \cdot 5}{\gcd(153 \cdot 5)} = \frac{153 \cdot 5}{1} = 765$

$\boxed{o(8, 6, 4) = 765}$

(Note: $\text{lcm}(a, b)$ is the least common multiple of a and b. $\text{lcm}(a, b) = \frac{ab}{\gcd(a,b)}$)

(note also: $\text{lcm}(a, b, c) = \text{lcm}(\text{lcm}(a, b), c)$)

(Note: $o(m)$ in $\mathbb{Z}_n$ is given by $\frac{n}{\gcd(m,n)}$)

17. Suppose that $(A, *) \leq (G, *)$ and that $(B, *) \leq (H, *)$. Show that $(A \times B, *) \leq (G \times H, *)$.

Note: The operation $*$ in $(G, *)$ is the operator that acts on the first component of $(G \times H, *)$

The operation $*$ in $(H, *)$ is the operator that acts on the second component of $(G \times H, *)$

Also: Given $a_1, a_2 \in A$, the operation $*$ in $(A, *)$ assigns the same element to $a_1 * a_2$ that it assigns to $a_1 * a_2$ as elements of $(G, *)$

Given $b_1, b_2 \in B$, the operation $*$ in $(B, *)$ assigns the same element to $b_1 * b_2$ that it assigns to $b_1 * b_2$ as elements of $(H, *)$

OK, here's the proof:

1. First note that the operation $*$ on $A \times B$ is **closed** on $A \times B$.

   Since A and B are subgroups of A and B respectively, They are closed under their respective operations.

   i.e., Given $a_1, a_2 \in A$, $a_1 * a_2 \in A$, and given $b_1, b_2 \in B$, $b_1 * b_2 \in B$.

   Thus, given $(a_1, b_1); (a_2, b_2) \in A \times B$, $(a_1, b_1) * (a_2, b_2) = (a_1 * a_2, b_1 * b_2) \in A \times B$.

2. Next note that (e_A, e_B) is the identity of $(A \times B, *)$. This is shown below.

   $(a, b) * (e_A, e_B) = (a * e_A, b * e_B) = (a, b)$

   $(e_A, e_B) * (a, b) = (e_A * a, e_B * b) = (a, b)$

3. Given $(a, b) \in A \times B$, the inverse of (a, b) is $\left(a^{-1}, b^{-1}\right)$, where a^{-1} is the inverse of a in $(A, *_A)$ and b^{-1} is the inverse of b in $(B, *_B)$. This is shown below.

   $\left(a^{-1}, b^{-1}\right) * (a, b) = \left(a^{-1} * a, b^{-1} * b\right) = (e_A, e_B)$

   $(a, b) * \left(a^{-1}, b^{-1}\right) = \left(a * a^{-1}, b * b^{-1}\right) = (e_A, e_B)$

4. The operation $*$ on $A \times B$ is associative. This follows directly from the fact that this same operation is associative on $G \times H$, of which $A \times B$ is a subset.$\blacksquare$