# MTH 1126 Test #3 (9am Class) - Solutions

Spring 2022

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**Show CLEARLY how you arrive at your answers.**

1. $\int \sin^2(x)\,dx =$

   We have $\sin(x)$ raised to an even power and no factor of $\cos(x)$. U-sub won't work here. We have to express $\sin^2(x)$ in a form other than $\sin(x)$ raised to a power.

   We can do this by using the identity: $\sin^2(x) = \frac{1-\cos(2x)}{2}$

   Thus, $\int \sin^2(x)\,dx = \int \frac{1-\cos(2x)}{2}dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx = \int \frac{1}{2}dx - \frac{1}{2}\int \underbrace{\cos(2x)}_{\cos(u)}\underbrace{dx}_{\frac{1}{2}du}$

   $= \frac{1}{2}x - \frac{1}{2}\left(\frac{1}{2}\cos(u)\right) + C = \frac{1}{2}x - \frac{1}{2}\left(\frac{1}{2}\sin(2x)\right) + C = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$

   $\boxed{\int \sin^2(x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C}$

2. $\lim_{x\to 0} \frac{\tan(x)}{x} =$

   $\lim_{x\to 0} \frac{\tan(x)}{x} \sim \frac{0}{0}$ (Use L'Hopital's Rule)

   $\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}[\tan(x)]}{\frac{d}{dx}[x]} = \lim_{x\to 0} \frac{\sec^2(x)}{1} = \frac{1}{1} = 1$

   $\boxed{\lim_{x\to 0} \frac{\tan(x)}{x} = 1}$

3. $\int \frac{x+23}{x^2-3x-10}dx =$

   Note that $\int \frac{x+23}{x^2-3x-10}dx$ does not fit the form: $\int \frac{1}{u}du$

   $u = x^2 - 3x - 10$
   $\frac{du}{dx} = 2x - 3$
   $du = (2x-3)\,dx$

   $\int \frac{x+23}{x^2-3x-10}dx = \int \underbrace{\frac{1}{x^2-3x-10}}_{\frac{1}{u}}\underbrace{(x+23)\,dx}_{\text{NOT a constant multiple of } du}$

   Therefore, we decompose $\frac{x+23}{x^2-3x-10}$ into the sum of simpler quotients:

1. Make sure that deg(numerator) $\leq$ deg(denominator)

2. Factor the denominator.

$\frac{x+23}{x^2-3x-10} = \frac{x+23}{(x-5)(x+2)}$

3. For each linear factor $(x+c)$, form the term $\frac{C_1}{x+c}$

$\frac{x+23}{x^2-3x-10} = \frac{x+23}{(x-5)(x+2)} = \frac{C_1}{x-5} + \frac{C_2}{x+2}$

4. Solve for the constants

$\frac{x+23}{(x-5)(x+2)} = \frac{C_1}{x-5} + \frac{C_2}{x+2}$

$\Rightarrow \frac{x+23}{(x-5)(x+2)}(x-5)(x+2) = \frac{C_1}{x-5}(x-5)(x+2) + \frac{C_2}{x+2}(x-5)(x+2)$

i.e., $x+23 = C_1(x+2) + C_2(x-5)$

Plu in "strategic values" of $x$ to find the values of the constants.

$\boxed{x=-2}$

$\Rightarrow 21 = -7C_2$

$\Rightarrow \boxed{C_2 = -3}$

$\boxed{x=5}$

$\Rightarrow 28 = 7C_1$

$\Rightarrow \boxed{C_1 = 4}$

Thus, $\frac{x+23}{x^2-3x-10} = \frac{C_1}{x-5} + \frac{C_2}{x+2} = \frac{4}{x-5} - \frac{3}{x+2}$

i.e., $\frac{x+23}{x^2-3x-10} = \frac{4}{x-5} - \frac{3}{x+2}$

Consequently, $\int \frac{x+23}{x^2-3x-10}dx = \int \left(\frac{4}{x-5} - \frac{3}{x+2}\right)dx = 4\int \frac{1}{x-5}dx - 3\int \frac{1}{x+2}dx = 4\ln|x-5| - 3\ln|x+2| + C$

$$\boxed{\int \frac{x+23}{x^2-3x-10}dx = 4\ln|x-5| - 3\ln|x+2| + C}$$

4. $\int \sin^3(x)\cos^7(x)\,dx =$

We have odd powers of $\sin(x)$ and $\cos(x)$.

Therefore, we can use either $\sin(x)$ or $\cos(x)$ as our "future du."

1. Reserve a factor of $\sin(x)$ to serve as our "future du."

$$= \int \sin^2(x)\cos^7(x)\underbrace{\sin(x)\,dx}_{\text{"future du"}}$$

This means that we intend to let $u = \cos(x)$

2. Convert remaining sines into cosines

$$= \int (1-\cos^2(x))\cos^7(x)\sin(x)\,dx = \int (\cos^7(x)-\cos^9(x))\sin(x)\,dx$$

$$\boxed{\begin{array}{ll} \text{Let} & u = \cos(x) \\ \Rightarrow & \frac{du}{dx} = -\sin(x) \\ \Rightarrow & du = -\sin(x)\,dx \\ \Rightarrow & -du = \sin(x)\,dx \end{array}}$$

$$= \int \underbrace{\left(\cos^7(x)-\cos^9(x)\right)}_{u^7-u^9}\underbrace{\sin(x)\,dx}_{-du} = \int (u^7-u^9)(-du) = \int (u^9-u^7)\,du = \frac{u^{10}}{10} - \frac{u^8}{8} + C$$

$$= \frac{\cos^{10}(x)}{10} - \frac{\cos^8(x)}{8} + C$$

$$\boxed{\int \sin^3(x)\cos^7(x)\,dx = \frac{\cos^{10}(x)}{10} - \frac{\cos^8(x)}{8} + C}$$

Alternative Solution appears on the next page

$\int \sin^3(x)\cos^7(x)\,dx =$

We have odd powers of $\sin(x)$ and $\cos(x)$.

Therefore, we can use either $\sin(x)$ or $\cos(x)$ as our "future du."

1. Reserve a factor of $\cos(x)$ to serve as our "future du."

$$= \int \sin^3(x)\cos^6(x)\underbrace{\cos(x)\,dx}_{\text{"future du"}}$$

This means that we intend to let $u = \sin(x)$

2. Convert remaining cosines into sines

$$= \int \sin^3(x)\left(\cos^2(x)\right)^3\underbrace{\cos(x)\,dx}_{\text{"future du"}}$$

$$= \int \sin^3(x)\left(1-\sin^2(x)\right)^3\cos(x)\,dx$$

$$= \int \sin^3(x)\left(-\sin^6(x)+3\sin^4(x)-3\sin^2(x)+1\right)\cos(x)\,dx$$

$$= \int \left(-\sin^9(x)+3\sin^7(x)-3\sin^5(x)+\sin^3(x)\right)\cos(x)\,dx$$

$$\boxed{\begin{array}{ll} \text{Let} & u = \sin(x) \\ \Rightarrow & \frac{du}{dx} = \cos(x) \\ \Rightarrow & du = \cos(x)\,dx \end{array}}$$

$$= \int \left(-u^9+3u^7-3u^5+u^3\right)du$$

$$= -\tfrac{1}{10}u^{10}+\tfrac{3}{8}u^8-\tfrac{1}{2}u^6+\tfrac{1}{4}u^4+C$$

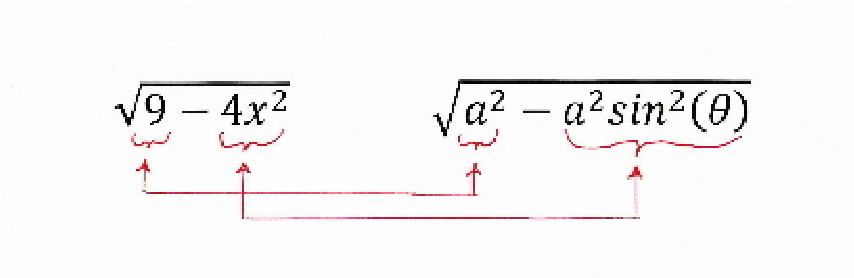
$$= -\tfrac{1}{10}\sin^{10}(x)+\tfrac{3}{8}\sin^8(x)-\tfrac{1}{2}\sin^6(x)+\tfrac{1}{4}\sin^4(x)+C$$

$$\boxed{\int \sin^3(x)\cos^7(x)\,dx = -\tfrac{1}{10}\sin^{10}(x)+\tfrac{3}{8}\sin^8(x)-\tfrac{1}{2}\sin^6(x)+\tfrac{1}{4}\sin^4(x)+C}$$

5. $\int x\sqrt{9-4x^2}dx =$ (Use Trig Substitution)

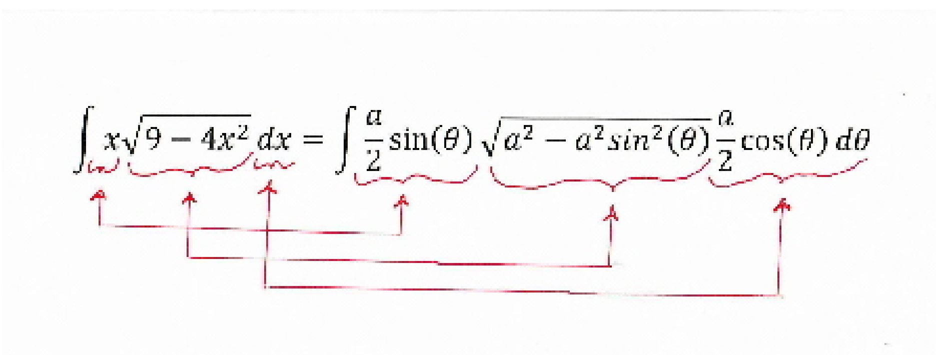
This can be done easily by U-substitution, but I think that we were instructed to do this using "Trig Substitution."

We match the radical $\sqrt{9-4x^2}$ with the radical $\sqrt{a^2 - a^2\sin^2(\theta)}$




| | | | |
|---|---|---|---|
| | $a^2$ | $=$ | $9$ |
| $\Rightarrow$ | $a$ | $=$ | $3$ |
| | $4x^2$ | $=$ | $a^2\sin^2(\theta)$ |
| $\Rightarrow$ | $2x$ | $=$ | $a\sin(\theta)$ |
| $\Rightarrow$ | $x$ | $=$ | $\frac{a}{2}\sin(\theta)$ |
| $\Rightarrow$ | $\frac{dx}{d\theta}$ | $=$ | $\frac{a}{2}\cos(\theta)$ |
| $\Rightarrow$ | $dx$ | $=$ | $\frac{a}{2}\cos(\theta)\,d\theta$ |

Rewrite the integral in terms of $\theta$



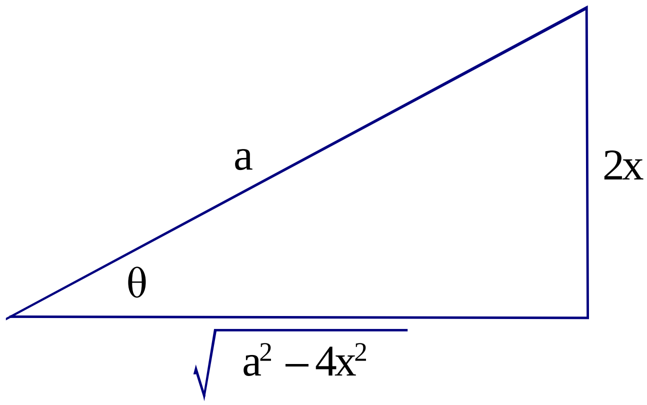

$$\int x\sqrt{9-4x^2}dx = \int \frac{a}{2}\sin(\theta)\sqrt{a^2 - a^2\sin^2(\theta)}\frac{a}{2}\cos(\theta)\,d\theta = \int \frac{a}{2}\sin(\theta)\sqrt{a^2\cos^2(\theta)}\frac{a}{2}\cos(\theta)\,d\theta$$

$$= \int \frac{a}{2}\sin(\theta)\,a\cos(\theta)\,\frac{a}{2}\cos(\theta)\,d\theta = \frac{a^3}{4}\int \underbrace{\cos^2(\theta)}_{u^2}\underbrace{\sin(\theta)\,d\theta}_{-du}$$

| | | | |
|---|---|---|---|
| | $u$ | $=$ | $\cos(\theta)$ |
| $\Rightarrow$ | $\frac{du}{d\theta}$ | $=$ | $-\sin(\theta)$ |
| $\Rightarrow$ | $du$ | $=$ | $-\sin(\theta)\,d\theta$ |
| $\Rightarrow$ | $-du$ | $=$ | $\sin(\theta)\,d\theta$ |

$$= \frac{a^3}{4}\int u^2(-du) = -\frac{a^3}{4}\int u^2du = -\frac{a^3}{4}\frac{u^3}{3} + C = -\frac{a^3}{4}\frac{\cos^3(\theta)}{3} + C$$

To convert back to $x$, recall that $x = \frac{a}{2}\sin(\theta)$ (i.e., $\sin(\theta) = \frac{2x}{a} = \frac{\text{opp}}{\text{hyp}}$)




$$\int x\sqrt{9-4x^2}dx = \ldots = -\frac{a^3}{4}\frac{\cos^3(\theta)}{3} + C = -\frac{a^3}{4}\frac{\left(\frac{\sqrt{a^2-4x^2}}{a}\right)^3}{3} + C = -\frac{a^3}{4}\frac{\frac{\left(a^2-4x^2\right)^{\frac{3}{2}}}{a^3}}{3} + C$$

$$= -\frac{1}{4}\frac{\left(a^2-4x^2\right)^{\frac{3}{2}}}{3} + C = -\frac{\left(9-4x^2\right)^{\frac{3}{2}}}{12} + C = -\frac{1}{12}\left(9-4x^2\right)^{\frac{3}{2}} + C$$

$$\boxed{\int x\sqrt{9-4x^2}dx = -\frac{1}{12}\left(9-4x^2\right)^{\frac{3}{2}} + C}$$

6. $\lim_{x\to 0} \frac{\ln(x+1)}{e^x-1} \sim \frac{0}{0}$

Use L'Hopital's Rule!

$$\lim_{x\to 0} \frac{\ln(x+1)}{e^x-1} = \lim_{x\to 0} \frac{\frac{d}{dx}[\ln(x+1)]}{\frac{d}{dx}[e^x-1]} = \lim_{x\to 0} \frac{\frac{1}{x+1}}{e^x} = \frac{\frac{1}{(0)+1}}{e^0} = 1$$

$$\boxed{\lim_{x\to 0} \frac{\ln(x+1)}{e^x-1} = 1}$$

7. $\int e^x \cos(x)\, dx =$ (Use Integration by Parts)

When doing integration by parts and the integrand consists of two functions that have derivatives that are "cyclic," the rule of thumb is that either function can be $u$ and either function can be $dv$. We repeat the process of Integration by Parts until we get the original integral. Then we solve for the integral algebraically.

Arbitrarily, we'll let $u = e^x$ and $dv = \cos(x)\, dx$

This yields:

| | | | | | | |
|---|---|---|---|---|---|---|
| $u$ | $=$ | $e^x$ | | $dv$ | $=$ | $\cos(x)\, dx$ |
| $\frac{du}{dx}$ | $=$ | $e^x$ | | $\int dv$ | $=$ | $\int \cos(x)\, dx$ |
| $du$ | $=$ | $e^x dx$ | | $v$ | $=$ | $\sin(x)$ |

Thus we have:

$$\int \underbrace{e^x}_{u} \underbrace{\cos(x)\, dx}_{dv} = \int u\, dv = uv - \int v du = e^x \sin(x) - \int \sin(x)\, e^x dx$$

Hmmm . . . it looks like we'll have to use Integration by Parts again. The "Rule of Thumb" when performing Integration by Parts multiple times is that we don't switch the roles of $u$ and $dv$. (e.g., if $u$ is an exponential the first time, then $u$ should be the exponential the second time. If $dv$ is the trig function the first time, then $dv$ should be the trig function the second time. Consequently, we have:

| | | | | | | |
|---|---|---|---|---|---|---|
| $u$ | $=$ | $e^x$ | | $dv$ | $=$ | $\sin(x)\, dx$ |
| $\frac{du}{dx}$ | $=$ | $e^x$ | | $\int dv$ | $=$ | $\int \sin(x)\, dx$ |
| $du$ | $=$ | $e^x dx$ | | $v$ | $=$ | $-\cos(x)$ |

Thus:

$$\int \underbrace{e^x}_{u} \underbrace{\cos(x)\, dx}_{dv} = e^x \sin(x) - \int \sin(x)\, e^x dx = e^x \sin(x) - \int \underbrace{e^x}_{u} \underbrace{\sin(x)\, dx}_{dv}$$

$$= e^x \sin(x) - \int u\, dv = e^x \sin(x) - \left[uv - \int v du\right]$$

$$= e^x \sin(x) - \left[ \underbrace{e^x}_{u} \underbrace{(-\cos(x))}_{v} - \int \underbrace{(-\cos(x))}_{v} \underbrace{e^x dx}_{du} \right] = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\, dx$$

i.e., $\int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\, dx$

We have established that:

$\int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\, dx$

We solve for $\int e^x \cos(x)\, dx$ algebraically:

$2 \int e^x \cos(x)\, dx = e^x \sin(x) + e^x \cos(x)$

$\Rightarrow \int e^x \cos(x)\, dx = \frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + C$

i.e., $\int e^x \cos(x)\, dx = \frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + C$

WOW! Extra (10 pts - all or nothing)

$\int x \arctan(x)\, dx =$

$\arctan(x)$ is a transcendental function whose derivative is algebraic. $\left(\frac{d}{dx}\left[\arctan(x)\right] = \frac{1}{1+x^2}\right)$

In such a case, we let the transcendental function be $u$.

Therefore, $\int \underbrace{\arctan(x)}_{u} \underbrace{x dx}_{dv} = \int u dv = uv - \int v du = \arctan(x)\left(\frac{1}{2}x^2\right) - \int \left(\frac{1}{2}x^2\right)\frac{1}{1+x^2} dx$

| | | | | | | |
|---|---|---|---|---|---|---|
| $u$ | $=$ | $\arctan(x)$ | | $dv$ | $=$ | $xdx$ |
| $\frac{du}{dx}$ | $=$ | $\frac{1}{1+x^2}$ | | $\int dv$ | $=$ | $\int xdx$ |
| $du$ | $=$ | $\frac{1}{1+x^2}dx$ | | $v$ | $=$ | $\frac{1}{2}x^2$ |

$= \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}\int \frac{x^2}{1+x^2} dx = \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}\int \left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right) dx$

$= \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx = \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}\left(x - \arctan(x)\right) + C$

$= \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}x + \frac{1}{2}\arctan(x) + C$

$$\boxed{\int x \arctan(x)\, dx = \frac{1}{2}x^2 \arctan(x) - \frac{1}{2}x + \frac{1}{2}\arctan(x) + C}$$