MTH 3311 Test #3 - Solutions

SPRING 2020

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Show CLEARLY how you arrive at your answers

1. Find the general solution of the equation: $y'' + 3y' - 4y = 2\sin(x) + 4\cos(x)$

First, find the solution to the complementary equation $y'' + 3y' - 4y = 0$

The auxiliary equation is $m^2 + 3m - 4 = 0$

$\Rightarrow m^2 + 3m - 4 = 0 \Rightarrow (m-1)(m+4) = 0 \Rightarrow m_1 = 1$ and $m_2 = -4$

$\Rightarrow y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{1x} + c_2 e^{-4x}$

i.e., $y_c = c_1 e^{x} + c_2 e^{-4x}$

For the particular solution, we imagine that $y_p = A\sin(x) + B\cos(x)$

$y_p' = A\cos(x) - B\sin(x)$

$y_p'' = -A\sin(x) - B\cos(x)$

To find A, B and C, we plug these into the original equation, $y'' + 3y' - 4y = 2\sin(x) + 4\cos(x)$

This yields:

$$\underbrace{(-A\sin(x) - B\cos(x))}_{y''} + \underbrace{3(A\cos(x) - B\sin(x))}_{3y'} - \underbrace{4(A\sin(x) + B\cos(x))}_{4y} = 2\sin(x) + 4\cos(x)$$

$\Rightarrow (-5A - 3B)\sin(x) + (3A - 5B)\cos(x) = 2\sin(x) + 4\cos(x)$

Comparing the coefficients of $\sin(x)$ and $\cos(x)$, we get:

$$\begin{array}{rcrcrl} -5A & - & 3B & = & 2 & \text{(Eq. 1)} \\ 3A & - & 5B & = & 4 & \text{(Eq. 2)} \\ \hline\hline & - & \frac{34}{5}B & = & \frac{26}{5} & \text{(Eq. 1 + Eq. 2)} \end{array}$$

$\Rightarrow B = -\frac{26}{34} = -\frac{13}{17}$

i.e. $B = -\frac{13}{17}$

Plugging $B = -\frac{13}{17}$ into Eq. 2, we get: $3A - 5\left(-\frac{13}{17}\right) = 4$

$\Rightarrow 3A + \left(\frac{65}{17}\right) = 4 \Rightarrow 3A + \frac{65}{17} = \frac{68}{17} \Rightarrow 3A + \frac{65}{17} = \frac{68}{17} \Rightarrow 3A = \frac{3}{17}$

$\Rightarrow A = \frac{1}{17}$

Hence, $y_p = \frac{1}{17}\sin(x) - \frac{13}{17}\cos(x)$

The solution to the original equation is: $y = y_p + y_c$

$$\boxed{\Rightarrow y = \frac{1}{17}\sin(x) - \frac{13}{17}\cos(x) + c_1 e^{x} + c_2 e^{-4x}}$$

2. Find the general solution of the equation: $y'' - 6y' + 9y = 4x^2 + 3x - 9$

First, find the solution to the complementary equation $y'' - 6y' + 9y = 0$

The auxiliary equation is $m^2 - 6m + 9 = 0$

$\Rightarrow m^2 - 6m + 9 = 0 \Rightarrow (m-3)(m-3) = 0 \Rightarrow m_1 = 3$ and $m_2 = 3$ (Double Root)

In this case, $y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{3x} + c_2 e^{3x}$ won't work, because the two terms are not independent solutions.

To remedy the situation, we multiply the second term by x.

This yields: $y_c = c_1 e^{3x} + c_2 x e^{3x}$

Since the Right Hand side of the original equation is $4x^2 + 3x - 9$, we imagine that:
$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

To find A, B and C, we plug these into the original equation: $y'' - 6y' + 9y = 4x^2 + 3x - 9$

This yields:

$$\underbrace{2A}_{y''} - \underbrace{6(2Ax + B)}_{6y'} + \underbrace{9\left(Ax^2 + Bx + C\right)}_{9y} = 4x^2 + 3x - 9$$

$\Rightarrow 9Ax^2 - (12A - 9B)x + (2A - 6B + 9C) = 4x^2 + 3x - 9$

Comparing the coefficients of the different powers of x, we get:

$$\begin{array}{lllllllll} 9A & & & & & = & 4 & & \text{(Eq. 1)} \\ -12A & + & 9B & & & = & 3 & & \text{(Eq. 2)} \\ 2A & - & 6B & + & 9C & = & -9 & & \text{(Eq. 3)} \end{array}$$

From Eq. 1, we get: $A = \frac{4}{9}$

Plugging $A = \frac{4}{9}$ into Eq. 2, we get: $-12\left(\frac{4}{9}\right) + 9B = 3 \Rightarrow 9B = 3 + \frac{48}{9} \Rightarrow 9B = \frac{75}{9} = \frac{25}{3} \Rightarrow B = \frac{25}{27}$

i.e., $B = \frac{25}{27}$

Plugging $A = \frac{4}{9}$ and $B = \frac{25}{27}$ into Eq. 3, we get: $2\left(\frac{4}{9}\right) - 6\left(\frac{25}{27}\right) + 9C = -9 \Rightarrow 9C = -9 - \frac{8}{9} + \frac{50}{9} \Rightarrow 9C = -\frac{13}{3}$

i.e., $C = -\frac{13}{27}$
Hence, $y_p = \frac{4}{9}x^2 + \frac{25}{27}x - \frac{13}{27}$

The solution to the original equation is: $y = y_p + y_c$

$$\boxed{\Rightarrow y = \frac{4}{9}x^2 + \frac{25}{27}x - \frac{13}{27} + c_1 e^{3x} + c_2 x e^{3x}}$$

3. Find the general solution of the equation: $y'' - 4y' + 20y = e^{3x}$

First, find the solution to the complementary equation $y'' - 4y' + 20y = 0$

The auxiliary equation is $m^2 - 4m + 20 = 0$

I don't think that I can factor this, so I'll use the quadrativ equation:

$m = \frac{-(-4)\pm\sqrt{(-4)^2-(4)(1)(20)}}{(2)(1)} = \frac{4\pm\sqrt{16-80}}{2} = \frac{4\pm\sqrt{-64}}{2} = \frac{4\pm 8i}{2} = 2 \pm 4i$

$m_1 = 2 + 4i$ and $m_2 = 2 - 4i$

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} = c_1 e^{(2+4i)x} + c_2 e^{(2-4i)x} = c_1 e^{2x+4ix} + c_2 e^{2x-4ix} = c_1 e^{2x} e^{4ix} + c_2 e^{2x} e^{-4ix}$

$= e^{2x}\left(c_1 e^{4ix} + c_2 e^{-4ix}\right) = e^{2x}\left(A\cos(4x) + B\sin(4x)\right)$

i.e., $y_c = e^{2x}\left(A\cos(4x) + B\sin(4x)\right)$

Next, we find the particular solution.

Since the Right Hand side of the original equation is e^{3x}, we suspect that:

$y_p = Ae^{3x}$

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

Plugging these into the equation $y'' - 4y' + 20y = e^{3x}$ we have:

$\underbrace{\left(9Ae^{3x}\right)}_{y''} - \underbrace{4\left(3e^{3x}\right)}_{4y'} + \underbrace{20\left(e^{3x}\right)}_{20y} = e^{3x}$

$\Rightarrow 17A = 1$

$\Rightarrow A = \frac{1}{17}$

$\Rightarrow y_p = \frac{1}{17}e^{3x}$

Our general solution is $y = y_p + y_c = \frac{1}{17}e^{3x} + A\cos(x) + B\sin(x)$

$$\boxed{y = \frac{1}{17}e^{3x} + e^{2x}\left(A\cos(4x) + B\sin(4x)\right)}$$

4. Find the general solution of the equation: $4y'' - 4y' + y = e^{\frac{1}{2}x} \ln(x)$

First, we find the solution to the complementary equation $4y'' - 4y' + y = 0$

The auxiliary equation is $4m^2 - 4m + 1 = 0$

$4m^2 - 4m + 1 = 0 \Rightarrow (2m-1)(2m-1) \Rightarrow m_1 = m_2 = \frac{1}{2}$ (Double Root)

To remedy the situation, we multiply the second term by x.

$\Rightarrow y_c = c_1 e^{m_1 x} + c_2 x e^{m_2 x} = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$

i.e., $y_c = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$

Next, we find the particular solution.

Since the right hand side of the original equation is not a linear combination of sines, cosines, exponentials, and polynomials, the method of Undetermined Coefficients won't work.

Therefore we must use Variation of Parameters.

We convert the complementary solution into the general solution: $y = A(x) e^{\frac{1}{2}x} + B(x) x e^{\frac{1}{2}x}$

We have two restrictions that we can impose on this pair of functions.

The first restriction that we impose is that the pair $A(x), B(x)$ actually does make the equation

$y = A(x) e^{\frac{1}{2}x} + B(x) x e^{\frac{1}{2}x}$ the general solution.

$$y = A(x) e^{\frac{1}{2}x} + B(x) x e^{\frac{1}{2}x}$$

$$\Rightarrow y' = A(x)\left(\frac{1}{2}e^{\frac{1}{2}x}\right) + A'(x) e^{\frac{1}{2}x} + B'(x) x e^{\frac{1}{2}x} + B(x) e^{\frac{1}{2}x} + B(x)\frac{1}{2} x e^{\frac{1}{2}x}$$

To simplify this expression, we impose our **second restriction:**

$$A'(x) e^{\frac{1}{2}x} + B'(x) x e^{\frac{1}{2}x} = 0 \qquad \text{(Eq.1)}$$

$$\Rightarrow y' = A(x)\left(\frac{1}{2}e^{\frac{1}{2}x}\right) + B(x) e^{\frac{1}{2}x} + B(x)\frac{1}{2} x e^{\frac{1}{2}x}$$

$$\Rightarrow y'' = A(x)\left(\frac{1}{4}e^{\frac{1}{2}x}\right) + A'(x)\left(\frac{1}{2}e^{\frac{1}{2}x}\right) + B'(x) e^{\frac{1}{2}x} + \frac{1}{2}B(x) e^{\frac{1}{2}x} + \frac{1}{2}B'(x) x e^{\frac{1}{2}x} + \frac{1}{2}B(x) e^{\frac{1}{2}x}$$
$$+\frac{1}{4}B(x) x e^{\frac{1}{2}x}$$

Rearranging the terms, we have:

$$y'' = A(x)\left(\frac{1}{4}e^{\frac{1}{2}x}\right) + B'(x) e^{\frac{1}{2}x} + \frac{1}{2}B(x) e^{\frac{1}{2}x} + \frac{1}{2}B(x) e^{\frac{1}{2}x} + \frac{1}{4}B(x) x e^{\frac{1}{2}x} + \frac{1}{2}\underbrace{\left(A'(x) e^{\frac{1}{2}x} + B'(x) x e^{\frac{1}{2}x}\right)}_{=0 \text{ (by Eq, 1)}}$$

i.e., $y'' = A(x)\left(\frac{1}{4}e^{\frac{1}{2}x}\right) + B'(x) e^{\frac{1}{2}x} + B(x) e^{\frac{1}{2}x} + \frac{1}{4}B(x) x e^{\frac{1}{2}x}$

We plug these into the original equation: $4y'' - 4y' + y = e^{\frac{1}{2}x} \ln(x)$

$$\begin{array}{rcrcrcrcrcr}
4y'' & = & A(x)\,e^{\frac{1}{2}x} & + & 4B'(x)\,e^{\frac{1}{2}x} & + & 4B(x)\,e^{\frac{1}{2}x} & + & B(x)\,xe^{\frac{1}{2}x} & & \\
-4y' & = & -2A(x)\,e^{\frac{1}{2}x} & & & - & 4B(x)\,e^{\frac{1}{2}x} & - & 2B(x)\,xe^{\frac{1}{2}x} & & \\
y & = & A(x)\,e^{\frac{1}{2}x} & & & & & + & B(x)\,xe^{\frac{1}{2}x} & & \\
\hline\hline
4y''-4y'+y & = & & & 4B'(x)\,e^{\frac{1}{2}x} & & & & & = & e^{\frac{1}{2}x}\ln(x)
\end{array}$$

Thus, we have: $4B'(x)\,e^{\frac{1}{2}x} = e^{\frac{1}{2}x}\ln(x)$

$\Rightarrow B'(x) = \frac{1}{4}\ln(x)$

$\Rightarrow B(x) = \frac{1}{4}\int \ln(x)\,dx = \frac{1}{4}(x\ln(x) - x) + C_2$

To find $A(x)$, we refer to Eq. 1

$\Rightarrow A'(x)\,e^{\frac{1}{2}x} + B'(x)\,xe^{\frac{1}{2}x} = 0$

$A'(x) + B'(x)\,x = 0$

$\Rightarrow A'(x) + \frac{1}{4}\ln(x)\,x = 0$

$\Rightarrow A'(x) = -\frac{1}{4}x\ln(x)$

$\Rightarrow A(x) = -\frac{1}{4}\int x\ln(x)\,dx = \frac{1}{16}x^2 - \frac{1}{8}x^2\ln x + C_1$ (using Integration by Parts)

i.e. $A(x) = \frac{1}{16}x^2 - \frac{1}{8}x^2\ln x + C_1$

Recall: Our general solution is: $y = A(x)\,e^{\frac{1}{2}x} + B(x)\,xe^{\frac{1}{2}x}$

$y = \left(\frac{1}{16}x^2 - \frac{1}{8}x^2\ln x + C_1\right)e^{\frac{1}{2}x} + \left(\frac{1}{4}(x\ln(x) - x) + C_2\right)xe^{\frac{1}{2}x}$

Simplifying, we have:

$$\boxed{y = -\frac{3}{16}x^2e^{\frac{1}{2}x} + \frac{1}{8}x^2\ln(x)\,e^{\frac{1}{2}x} + C_1e^{\frac{1}{2}x} + C_2xe^{\frac{1}{2}x}}$$

5. Find the general solution of the equation: $x^2y'' - xy' - 8y = 3x^3 - 2x + 5$

First, find the solution to the complementary equation $x^2y'' - xy' - 8y = 0$

Our strategy is to seek solutions of the form:

$$y = x^\lambda$$

$$\Rightarrow y' = \lambda x^{\lambda-1}$$

$$\Rightarrow y'' = \lambda(\lambda - 1)x^{\lambda-2} = \left(\lambda^2 - \lambda\right)x^{\lambda-2}$$

Plugging these into the complementary equation $x^2y'' - xy' - 8y = 0$, we have:

$$x^2\left(\lambda^2 - \lambda\right)x^{\lambda-2} - x\lambda x^{\lambda-1} - 8x^\lambda = 0$$

$$\Rightarrow \left(\lambda^2 - \lambda\right)x^\lambda - \lambda x^\lambda - 8x^\lambda = 0$$

$$\Rightarrow \left(\lambda^2 - \lambda\right) - \lambda - 8 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 8 = 0$$

$$\Rightarrow (\lambda + 2)(\lambda - 4) = 0$$

$$\Rightarrow \lambda_1 = -2;\ \lambda_2 = 4$$

Our complementary solution is:

$$y_c = c_1x^{\lambda_1} + c_2x^{\lambda_2} = c_1x^{-2} + c_2x^4$$

Next, we find our particular solution

Since the right hand side of the equation is the polynomial $3x^3 - 2x + 5$, we guess that the particular solution is a polynomial having only terms of the same degree that appear on the right hand side of the original equation.

Thus, we guess that:

$$y = Ax^3 + Bx + C$$

$$\Rightarrow y' = 3Ax^2 + B$$

$$\Rightarrow y'' = 6Ax$$

To find A, B and C, we plug these into the original equation, $x^2y'' - xy' - 8y = 3x^3 - 2x + 5$.

This yields:

$$x^2(6Ax) - x\left(3Ax^2 + B\right) - 8\left(Ax^3 + Bx + C\right) = 3x^3 - 2x + 5$$

$$\Rightarrow (6 - 3 - 8)Ax^3 + (-1 - 8)Bx - (8)C = 3x^3 - 2x + 5$$

$$\Rightarrow -5A = 3 \Rightarrow A = -\tfrac{3}{5}$$

And $-9B = -2 \Rightarrow B = \frac{2}{9}$

And $-8C = 5 \Rightarrow C = -\frac{5}{8}$

Hence, $y_p = -\frac{3}{5}x^3 + \frac{2}{9}x - \frac{5}{8}$

The solution to the original equation is: $y = y_p + y_c$

$$\boxed{\Rightarrow y = -\frac{3}{5}x^3 + \frac{2}{9}x - \frac{5}{8} + c_1 x^{-2} + c_2 x^4}$$