MTH 1125 (11 am) Test #3

Fall 2019

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Instructions. Show CLEARLY how you arrive at your answers.

1. $f(x) = x^3 - 3x - 5$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

   i. Compute $f'(x)$ and find the critical numbers

   $f'(x) = 3x^2 - 3$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 0$ and solve for x

   $\Rightarrow f'(x) = 3x^2 - 3 = 0$

   $\Rightarrow 3x^2 - 3 = 0$

   $\Rightarrow x^2 - 1 = 0$

   $\Rightarrow (x+1)(x-1) = 0$

   $\Rightarrow x = -1$ and $x = 1$ are critical numbers.

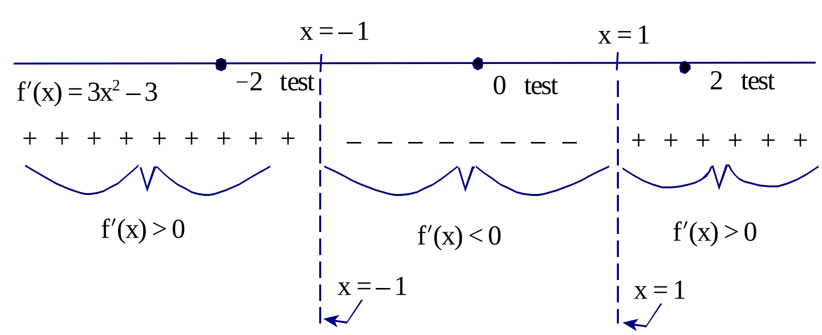
   "Type b" $(f'(c)$ is undefined)

   Look for x-value that causes division by zero.

   No "type b" critical numbers

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

3. Pick a "test point" from each interval to plug into $f'(x)$



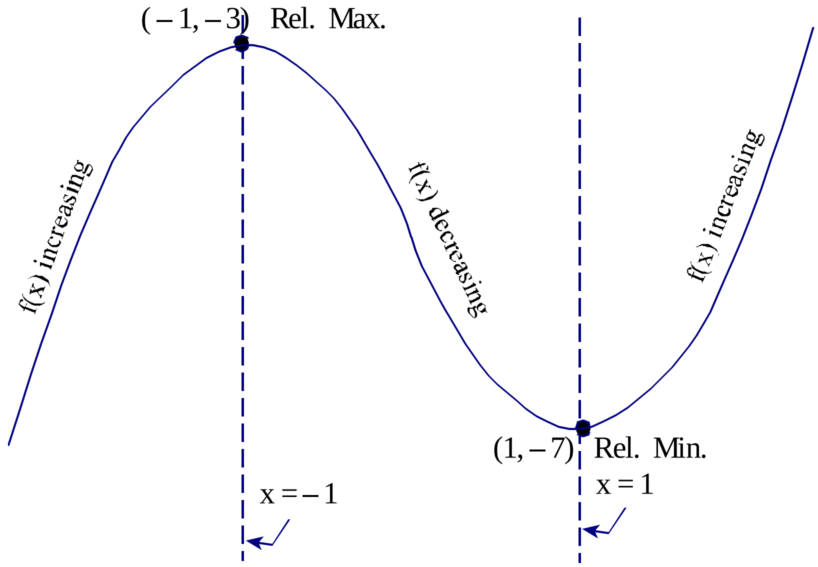
$f(x)$ is **increasing** on the interval(s) $(-\infty, -1)$ and $(1, \infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-1, 1)$

(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(-1, f(-1)) = (-1, -3)$

Rel Min $(1, f(1)) = (1, -7)$

2. $f(x) = x^4 + 2x^3 - 12x^2 - 6x + 3$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection.

   1. Compute $f''(x)$ and find possible points of inflection.

      $f'(x) = 4x^3 + 6x^2 - 24x - 6$

      $f''(x) = 12x^2 + 12x - 24$

      Find possible points of inflection:

      a. "Type a" ($f''(x) = 0$)

         Set $f''(x) = 0$

         $\Rightarrow f''(x) = 12x^2 + 12x - 24 = 0$

         $\Rightarrow x^2 + x - 2 = 0$

         $\Rightarrow (x+2)(x-1) = 0$

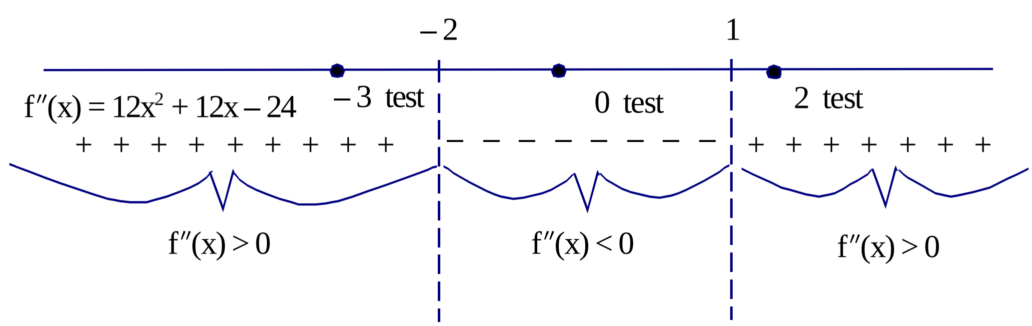
         $x = -2, 1$ possible "type a" points of inflection

      b. "Type b" ($f''(x)$ undefined)

         No "Type b" points of inflection

   2. Draw a "sign graph" of $f''(x)$, using the possible points of inflection to partition the x-axis.

   3. Select a test point from each interval and plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-\infty, -2)$ and $(1, \infty)$
(because $f''(x)$ is positive on these intervals)

$f(x)$ is **concave down** on the interval $(-2, 1)$
(because $f''(x)$ is negative on this interval)

Since $f(x)$ changes concavity at $x = -2$ and $x = 1$, the points:
$(-2, f(-2)) = (-2, -33)$
and
$(1, f(1)) = (1, -12)$ **are** points of inflection.

3. $f(x) = 2x^3 - 9x^2 - 24x + 2$ on the interval $[-2, 2]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

Note: [1] $f(x)$ is continuous (since it is a polynomial) on the [2]closed, [3]finite interval $[-2, 2]$. Therefore, we can use the Absolute Max/Min Value Test.

i. Compute $f'(x)$ and find the critical numbers.

$f'(x) = 6x^2 - 18x - 24$

a. "Type a" ($f'(x) = 0$)

$f'(x) = 6x^2 - 18x - 24 = 0$

$\Rightarrow x^2 - 3x - 4 = 0$

$\Rightarrow (x+1)(x-4) = 0$

$\Rightarrow x = -1, 4$ are "type a" critical numbers

Since $x = 4$ is not in the interval $[-2, 2]$, we discard it as a critical number.

b. "Type b" ($f'(x)$ is undefined)

No "Type b" critical numbers

ii. Plug endpoints and critical numbers into $f(x)$ (the *original* function)

$f(-2) = 2(-2)^3 - 9(-2)^2 - 24(-2) + 2 = -2$

$f(-1) = 2(-1)^3 - 9(-1)^2 - 24(-1) + 2 = 15$ $\leftarrow$ Abs Max Value

$f(2) = 2(2)^3 - 9(2)^2 - 24(2) + 2 = -66$ $\leftarrow$ Abs Min Value

The Abs Max Value is 15 (attained at $x = -1$)
The Abs Min Value is -66 (attained at $x = 2$)

4. $f(x) = \frac{2}{7}x^{\frac{14}{5}} - x^{\frac{4}{5}}$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums. $\frac{2}{7}x^{\frac{14}{5}} - x^{\frac{4}{5}}$

1. Compute $f'(x)$ and find the critical numbers

   $f'(x) = \frac{28}{35}x^{\frac{9}{5}} - \frac{4}{5}x^{-\frac{1}{5}} = \frac{28x^{\frac{9}{5}}}{35} - \frac{4}{5x^{\frac{1}{5}}} = \frac{28x^{\frac{9}{5}}}{35}\frac{x^{\frac{1}{5}}}{x^{\frac{1}{5}}} - \frac{4}{5x^{\frac{1}{5}}}\frac{7}{7} = \frac{28x^2-28}{35x^{\frac{1}{5}}}$

   i.e., $f'(x) = \frac{28x^2-28}{35x^{\frac{1}{5}}}$

   a. "Type a" $(f'(c) = 0)$

      Set $f'(x) = 0$ and solve for x

      $\Rightarrow f'(x) = \frac{28x^2-28}{35x^{\frac{1}{5}}} = 0$

      $\Rightarrow 28x^2 - 28 = 0$

      $\Rightarrow x^2 - 1 = 0$

      $\Rightarrow (x+1)(x-1) = 0$

      $\Rightarrow x = -1$ and $x = 1$ are critical numbers.

   b. "Type b" $(f'(c)$ is undefined)

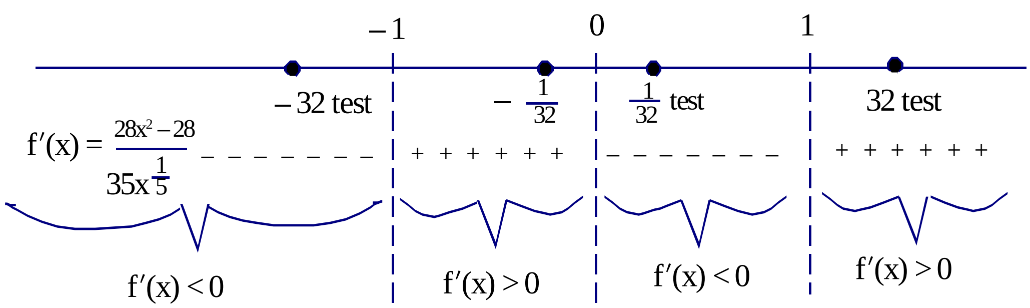
      Look for x-value that causes division by zero.

      $\Rightarrow 35x^{\frac{1}{5}} = 0$

      $\Rightarrow x = 0$ "type b" critical number

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

3. Pick a "test point" from each interval to plug into $f'(x)$



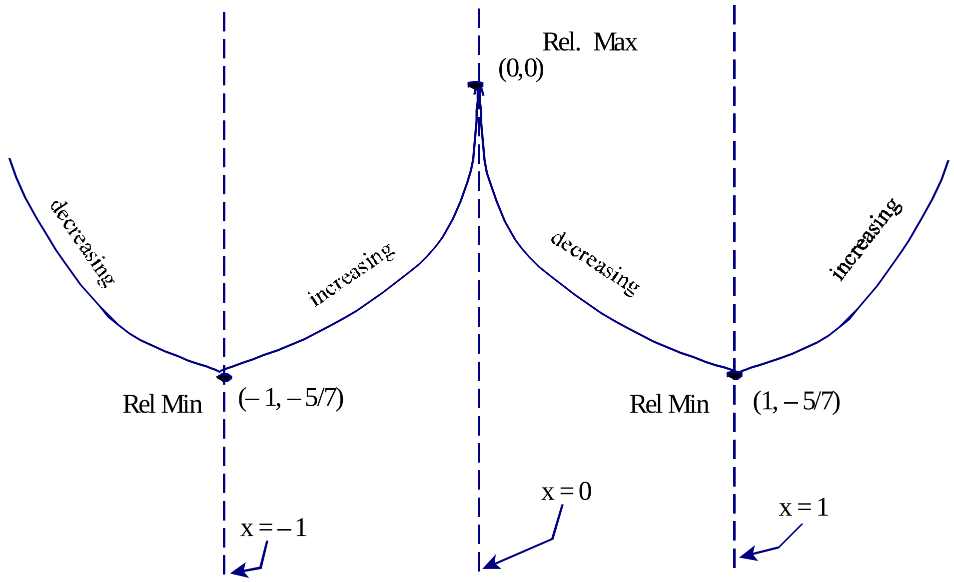
$f(x)$ is **increasing** on the interval(s) $(-1, 0)$ and $(1, \infty)$

(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(-\infty, -1)$ and $(0, 1)$

(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.

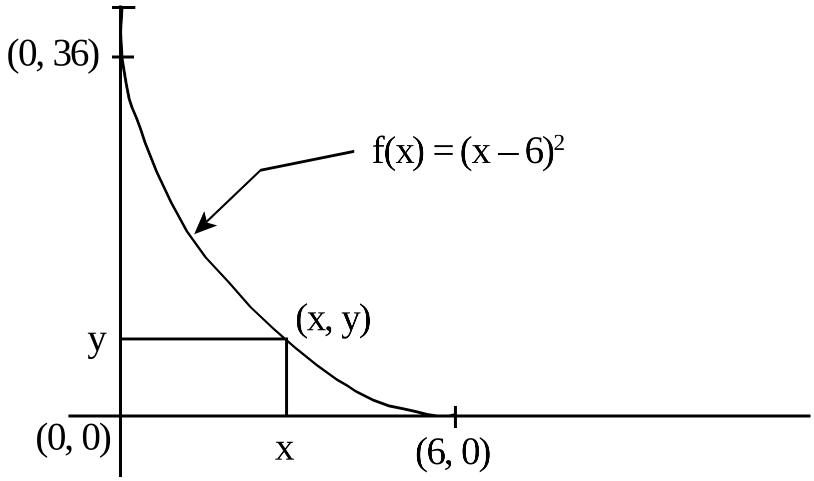


Rel Minimums: $(-1, f(-1)) = \left(-1, -\frac{5}{7}\right)$

and $(1, f(1)) = \left(1, -\frac{5}{7}\right)$

Rel Maximum: $(0, f(0)) = (0, 0)$

5. A rectangle is inscribed in the region bounded by the positive x-axis, the positive y-axis, and the graph of $f(x) = (x-6)^2$ as shown below. Determine the value of x that makes the area of the rectangle as large as possible.



1. Determine the quantity to be maximized - Give it a name!

   Maximize the **Area** of the rectangle, $A = xy$

   a. Draw a picture where relevant.

   1. 1. (Done)

2. Express A as a function of one other variable. (Use a restriction stated in the problem.)

   The restriction mentioned is that the point (x, y) must be on the graph of $f(x) = (x-6)^2$.

   Hence, the y-coordinate of the point (x, y) is $y = (x-6)^2$.

   Plug this into the equation $A = xy$

   $\Rightarrow A(x) = x(x-6)^2 = x^3 - 12x^2 + 36x$

   i.e., $A(x) = x^3 - 12x^2 + 36x$

3. Determine the restrictions on the independent variable x.

   From the picture, $0 \leq x \leq 6$

4. Maximize $A(x)$, using the techniques of Calculus.

   Note that $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0, 6]$.

   Thus, we can use the Absolute Max/Min Value Test.

   1. Find the critical numbers.

      $A'(x) = 3x^2 - 24x + 36$

      a. "Type a" $(f'(c) = 0)$

         $\Rightarrow A'(x) = 3x^2 - 24x + 36 = 0$

         $\Rightarrow 3x^2 - 24x + 36 = 0$

         $\Rightarrow x^2 - 8x + 12 = 0$

         $\Rightarrow (x-2)(x-6) = 0$

         $\Rightarrow x = 2$ and $x = 6$ are critical numbers

      b. "Type b" ($f'(c)$ is undefined)

         Look for x-values that cause division by zero in $f'(x)$

         (None)

   2. Plug critical numbers and endpoints into the *original* function.

      $A(0) = (0)^3 - 12(0)^2 + 36(0) = 0$

      $A(2) = (2)^3 - 12(2)^2 + 36(2) = 32 \longleftarrow$ Abs Max Value

      $A(6) = (6)^3 - 12(6)^2 + 36(6) = 0$

5. Make sure that we've answered the original question.

   "Determine the value of x ..."

   $\boxed{x = 2}$