MTH 1125 (12 pm) Test #3 - Solutions

FALL 2023

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Instructions. Show CLEARLY how you arrive at your answers.

1. $f(x) = 2x^3 - 3x^2 + 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums. (Caution - there are **two** critical numbers. Make sure you get them both!)

   i. Compute $f'(x)$ and find the critical numbers

   $f'(x) = 6x^2 - 6x$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 0$ and solve for x

   $\Rightarrow f'(x) = 6x^2 - 6x = 0$

   $\Rightarrow 6x(x-1) = 0$

   $\Rightarrow 6x = 0 \quad \text{or} \quad (x-1) = 0$

   $\Rightarrow x = 0$ and $x = 1$ are critical numbers.

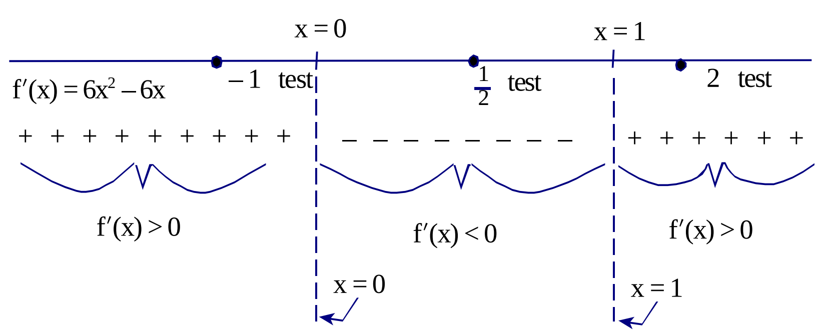
   b. "Type b" ($f'(c)$ is undefined)

   Look for x-value that causes division by zero.

   No "type b" critical numbers

2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

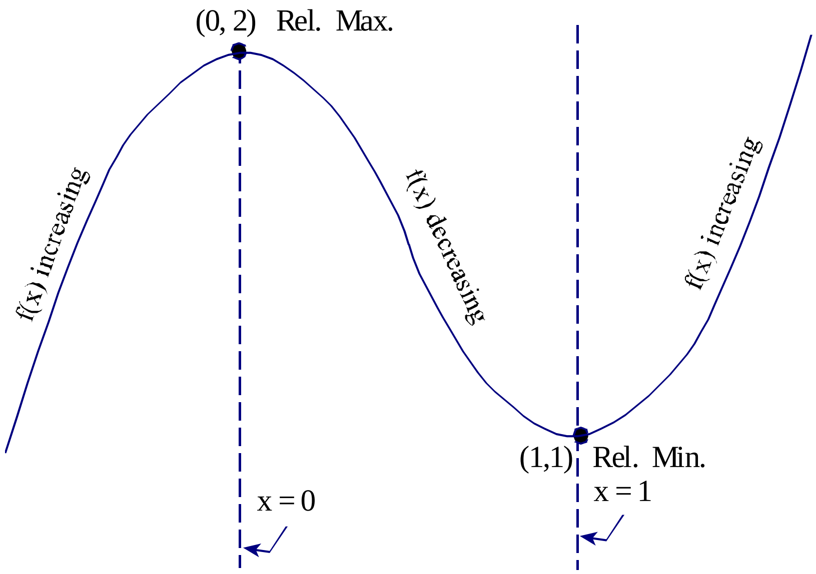
3. Pick a "test point" from each interval to plug into $f'(x)$



$f(x)$ is **increasing** on the interval(s) $(-\infty, 0)$ and $(1, \infty)$
(because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval(s) $(0, 1)$
(because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



Rel Max $(0, f(0)) = (0, 2)$

Rel Min $(1, f(1)) = (1, 1)$

2. $f(x) = x^4 - 8x^3 - 30x^2 + 6x + 3$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection. Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection.

   1. Compute $f''(x)$ and find possible points of inflection.

      $f'(x) = 4x^3 - 24x^2 - 60x + 6$

      $f''(x) = 12x^2 - 48x - 60$

      Find possible points of inflection:

      a. "Type a" $(f''(x) = 0)$

         Set $f''(x) = 0$

         $\Rightarrow f''(x) = 12x^2 - 48x - 60 = 0$

         $\Rightarrow x^2 - 4x - 5 = 0$

         $\Rightarrow (x+1)(x-5) = 0$

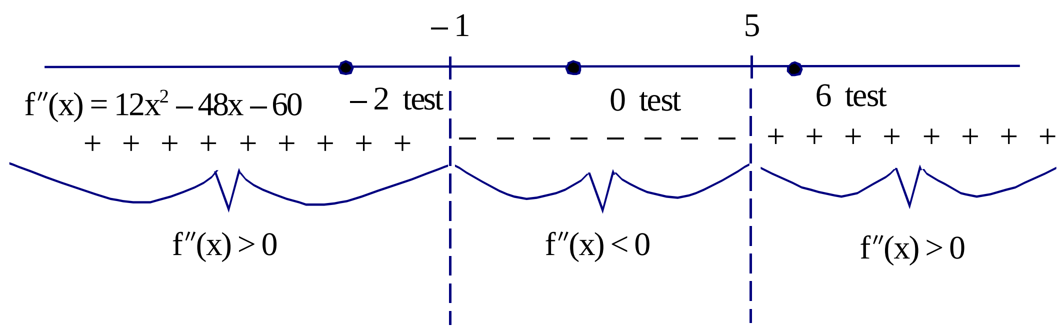
         $x = -1, 5$ possible "type a" points of inflection

      b. "Type b" $(f''(x)$ undefined$)$

         No "Type b" points of inflection

   2. Draw a "sign graph" of $f''(x)$, using the possible points of inflection to partition the x-axis.

   3. Select a test point from each interval and plug into $f''(x)$



$f(x)$ is **concave up** on the intervals $(-\infty, -1)$ and $(5, \infty)$
(because $f''(x)$ is positive on these intervals)

$f(x)$ is **concave down** on the interval $(-1, 5)$
(because $f''(x)$ is negative on this interval)

Since $f(x)$ changes concavity at $x = -1$ and $x = 5$, the points:
$(-1, f(-1)) = (-1, -24)$
and
$(5, f(5)) = (5, -1092)$ **are** points of inflection.

3. $f(x) = x^3 + 6x^2 - 15x + 3$ on the interval $[-2, 2]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

   Note: [1] $f(x)$ is continuous (since it is a polynomial) on the [2]closed, [3]finite interval $[-2, 2]$. Therefore, we can use the Absolute Max/Min Value Test.

   i. Compute $f'(x)$ and find the critical numbers.

      $f'(x) = 3x^2 + 12x - 15$

      a. "Type a" ($f'(x) = 0$)

         $f'(x) = 3x^2 + 12x - 15 = 0$

         $\Rightarrow 3(x^2 + 4x - 5) = 0$

         $\Rightarrow (x + 5)(x - 1) = 0$

         $\Rightarrow x = -5, 1$ are "type a" critical numbers

         Since $x = -5$ is not in the interval $[-2, 2]$, we discard it as a critical number.

      b. "Type b" ($f'(x)$ is undefined)

         No "Type b" critical numbers

   ii. Plug endpoints and critical numbers into $f(x)$ (the *original* function)

      $f(-2) = (-2)^3 + 6(-2)^2 - 15(-2) + 3 = 49 \leftarrow$ Abs Max Value

      $f(1) = (1)^3 + 6(1)^2 - 15(1) + 3 = -5 \leftarrow$ Abs Min Value

      $f(2) = (2)^3 + 6(2)^2 - 15(2) + 3 = 5$

The Abs Max Value is 49 (attained at $x = -2$)
The Abs Min Value is -5 (attained at $x = 1$)

4. $f(x) = \frac{1}{7}x^{\frac{14}{5}} - 2x^{\frac{4}{5}} + 1$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

1. Compute $f'(x)$ and find the critical numbers

$$f'(x) = \frac{2}{5}x^{\frac{9}{5}} - \frac{8}{5}x^{-\frac{1}{5}} = \frac{2x^{\frac{9}{5}}}{5} - \frac{8}{5x^{\frac{1}{5}}} = \frac{2x^{\frac{9}{5}}}{5}\frac{x^{\frac{1}{5}}}{x^{\frac{1}{5}}} - \frac{8}{5x^{\frac{1}{5}}} = \frac{2x^2-8}{5x^{\frac{1}{5}}}$$

i.e., $f'(x) = \frac{2x^2-8}{5x^{\frac{1}{5}}}$

a. "Type a" $(f'(c) = 0)$

Set $f'(x) = 0$ and solve for x

$\Rightarrow f'(x) = \frac{2x^2-8}{5x^{\frac{1}{5}}} = 0$

$\Rightarrow 2x^2 - 8 = 0$

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x+2)(x-2) = 0$

$\Rightarrow x = -2$ and $x = 2$ are critical numbers.

b. "Type b" $(f'(c)$ is undefined)

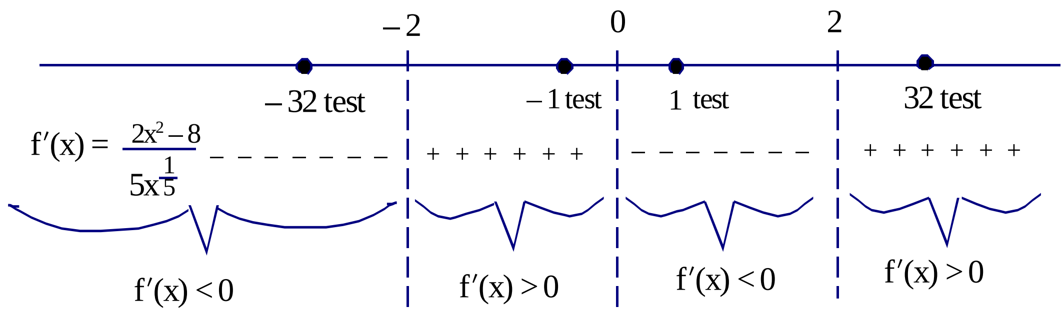
Look for x-value that causes division by zero.

$\Rightarrow 5x^{\frac{1}{5}} = 0$

$\Rightarrow x = 0$ "type b" critical number

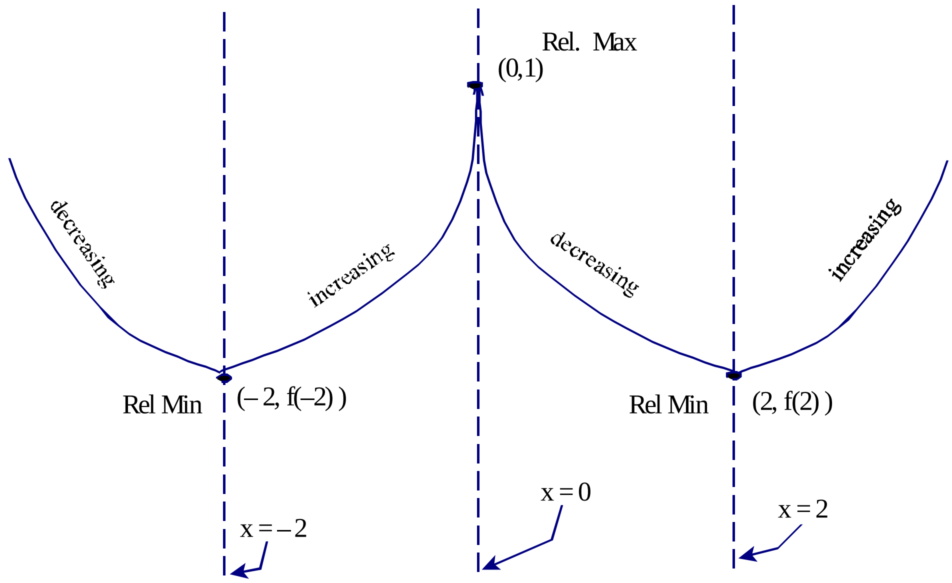
2. Draw a "sign graph" of $f'(x)$, using the critical numbers to partition the x-axis

3. Pick a "test point" from each interval to plug into $f'(x)$



> $f(x)$ is **increasing** on the interval(s) $(-2, 0)$ and $(2, \infty)$
>
> (because $f'(x)$ is positive on these intervals)
>
> $f(x)$ is **decreasing** on the interval(s) $(-\infty, -2)$ and $(0, 2)$
>
> (because $f'(x)$ is negative on that interval)

4. To find the relative maxes and mins, sketch a rough graph of $f(x)$.

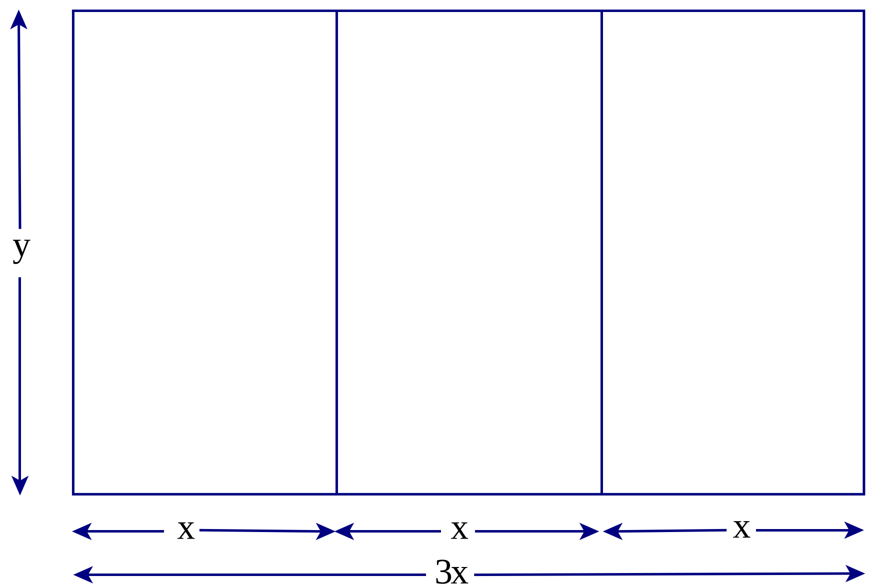


Rel Minimums: $(-2, f(-2)) = \left(-2, \frac{1}{7}(-2)^{\frac{14}{5}} - 2(-2)^{\frac{4}{5}} + 1\right)$

and $(2, f(2)) = \left(2, \frac{1}{7}(2)^{\frac{14}{5}} - 2(2)^{\frac{4}{5}} + 1\right)$

Rel Maximum: $(0, f(0)) = (0, 1)$

5. A rancher has 200 yards of fencing to enclose three adjacent rectangular corrals, as shown below. What overall dimensions should be used so that the enclosed area will be as large as possible?



Solution: Version 1 (Express area A as a function of x)

i. Declare what it is that you want to maximize/minimize — give it a name.

   Maximize enclosed area, $A = 3xy$

   Draw a picture

   (Done)

ii. Express A as a function of one variable. (Refer to a restriction stated in the problem.)

   Restriction: Use 200 ft of fencing

   $\Rightarrow 6x + 4y = 200$ yds

   $\Rightarrow 4y = 200$ yds $- 6x$

   $\Rightarrow y = 50$ yds $- \frac{3}{2}x$

   Plug this into the equation $A = 3xy$

   $\Rightarrow A = 3x\left(50 \text{ yds } - \frac{3}{2}x\right) = 150 \text{ yds } x - \frac{9}{2}x^2$

   i.e., $A(x) = 150 \text{ yds } x - \frac{9}{2}x^2$

iii. Determine the restrictions on the independent variable x.

$0 \text{ yds} \leq x \leq \frac{100}{3} \text{ yds}$

iv. Maximize/minimize the function, using the techniques of calculus

Since $A(x)$ is continuous on the closed, finite interval $\left[0 \text{ yds}, \frac{100}{3} \text{ yds}\right]$, we can use the Absolute Max/Min Value Test.

Compute $A'(x)$ and find the critical numbers

$A'(x) = 150 \text{ yds} - 9x$

Type a: set $A'(x) = 150 \text{ yds} - 9x = 0$

$\Rightarrow 9x = 150 \text{ yds}$

$\Rightarrow x = \frac{50}{3} \text{ yds}$ (critical number)

Type b: No Type b critical numbers

Plug endpoints and critical numbers into $A(x)$, the original function.

$A(0 \text{ yds}) = 150 \text{ yds}\,(0 \text{ yds}) - \frac{9}{2}(0 \text{ yds})^2 = 0 \text{ yds}^2$

$A\left(\frac{50}{3} \text{ yds}\right) = 150 \text{ yds}\left(\frac{50}{3} \text{ yds}\right) - \frac{9}{2}\left(\frac{50}{3} \text{ yds}\right)^2 = 1250 \text{ yds}^2 \leftarrow$ Abs Max Area

$A\left(\frac{100}{3} \text{ yds}\right) = 150 \text{ yds}\left(\frac{100}{3} \text{ yds}\right) - \frac{9}{2}\left(\frac{100}{3} \text{ yds}\right)^2 = 0 \text{ yds}^2$

v. Make sure we've answered the original question:

"What dimensions should be used . . . "

Length $= 3\text{x} = 3\left(\frac{50}{3}\text{yds}\right) = 50$ yds

Width $= y = 50 \text{ yds} - \frac{3}{2}x = 50 \text{ yds} - \frac{3}{2}\left(\frac{50}{3} \text{ yds}\right) = 25$ yds

$$\boxed{\begin{array}{l} \text{Length} = 3\text{x} = 50 \text{ yds} \\ \text{Width} = y = 25 \text{ yds} \end{array}}$$

Solution: Version 2 (Express area A as a function of y)

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Solution: Version 2 (Express area A as a function of y)

i. Declare what it is that you want to maximize/minimize — give it a name.

   Maximize enclosed area, $A = 3xy$

   Draw a picture

   (Done)

ii. Express A as a function of one variable. (Refer to a restriction stated in the problem.)

   Restriction: Use 200 ft of fencing

   $\Rightarrow 6x + 4y = 200$ yds

   $\Rightarrow 6x = 200$ yds $- \ 4y$

   $\Rightarrow 3x = 100$ yds $- \ 2y$

   Plug this into the equation $A = 3xy$

   $\Rightarrow A = (100 \text{ yds } - 2y)\, y = 100 \text{ yds } y - 2y^2$

   i.e., $A(x) = 100 \text{ yds } y - 2y^2$

iii. Determine the restrictions on the independent variable y.

   $0 \text{ yds} \leq y \leq 50 \text{ yds}$

iv. Maximize/minimize the function, using the techniques of calculus

   Since $A(y)$ is continuous on the closed, finite interval $[0 \text{ yds}, 50 \text{ yds}]$, we can use the Absolute Max/Min Value Test.

   Compute $A'(y)$ and find the critical numbers

   $A'(y) = 100 \text{ yds } -4y$

   Type a: set $A'(y) = 100 \text{ yds } -4y = 0$

   $\Rightarrow 4y = 100$ yds

   $\Rightarrow y = 25$ yds (critical number)

   Type b: No Type b critical numbers

Plug endpoints and critical numbers into $A(x)$, the original function.

$A(0 \text{ yds}) = 100 \text{ yds } (0 \text{ yds}) - 2(0 \text{ yds})^2 = 0 \text{ yds}^2$

$A(25 \text{ yds}) = 100 \text{ yds } (25 \text{ yds}) - 2(25 \text{ yds})^2 = 1250 \text{ yds}^2 \leftarrow$ Abs Max Area

$A(50 \text{ yds}) = 100 \text{ yds } (50 \text{ yds}) - 2(50 \text{ yds})^2 = 0 \text{ yds}^2$

v. Make sure we've answered the original question:

"What dimensions should be used . . . "

Width $= y = 25$ yds

Length $= 3\text{x} = 100$ yds $- 2y = 50$ yds

$$\boxed{\begin{array}{l} \text{Length} = 3\text{x} = 50 \text{ yds} \\ \text{Width} = y = 25 \text{ yds} \end{array}}$$