MTH 1125 11am Class - Test #4 - Solutions

FALL 2019

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Show CLEARLY how you arrive at your answers!

1. **Compute:** $\int \left(12x^5 + 9x^2 + 4x + 3 + 3\sqrt{x}\right) dx =$

$\int \left(12x^5 + 9x^2 + 4x + 3 + 3\sqrt{x}\right) dx = \int \left(12x^5 + 9x^2 + 4x + 3 + 3x^{\frac{1}{2}}\right) dx$

$= 12\left[\frac{x^6}{6}\right] + 9\left[\frac{x^3}{3}\right] + 4\left[\frac{x^2}{2}\right] + 3x + 3\left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right] + C = 2x^6 + 3x^3 + 2x^2 + 3x + 2x^{\frac{3}{2}} + C$

i.e., $\int \left(12x^5 + 9x^2 + 4x + 3 + 3\sqrt{x}\right) dx = 2x^6 + 3x^3 + 2x^2 + 3x + 2x^{\frac{3}{2}} + C$

2. **Compute:** $\int \left(9x^2+6x+2\right)^9 (3x+1)\,dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(9x^2+6x+2\right)^9$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(9x^2+6x+2\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(9x^2+6x+2\right)}_{\text{function}} - - - - \rightarrow \underbrace{(3x+1)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(9x^2+6x+2\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 9x^2+6x+2 \\ \Rightarrow \frac{du}{dx} &= 18x+6 \\ \Rightarrow du &= (18x+6)\,dx \\ \Rightarrow \frac{1}{6}du &= (3x+1)\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\left(9x^2+6x+2\right)^9}_{u^9}\underbrace{(3x+1)\,dx}_{\frac{1}{6}du} = \int u^9 \tfrac{1}{6}du = \tfrac{1}{6}\int u^9 du$$

4. Integrate (in terms of u).

$$\tfrac{1}{6}\int u^9 du = \tfrac{1}{6}\left[\tfrac{u^{10}}{10}\right] + C = \tfrac{1}{60}u^{10} + C$$

5. Re-express in terms of the original variable, x.

$$\int \left(9x^2+6x+2\right)^9 (3x+1)\,dx = \underbrace{\frac{1}{60}\left(9x^2+6x+2\right)^{10} + C}_{\frac{1}{60}u^{10}+C}$$

i.e., $\int \left(9x^2+6x+2\right)^9 (3x+1)\,dx = \frac{1}{60}\left(9x^2+6x+2\right)^{10} + C$

3. **Compute:** $\int \left(5\sin(x) + 4\sec^2(x) - 3\csc(x)\cot(x)\right) dx =$

$\int \left(5\sin(x) + 4\sec^2(x) - 3\csc(x)\cot(x)\right) dx = 5\left[-\cos(x)\right] + 4\left[\tan(x)\right] - 3\left[-\csc(x)\right] + C$

$= -5\cos(x) + 4\tan(x) + 3\csc(x) + C$

i.e., $\int \left(5\sin(x) + 4\sec^2(x) - 3\csc(x)\cot(x)\right) dx = -5\cos(x) + 4\tan(x) + 3\csc(x) + C$

4. **Compute:** $\int \cos\left(10x^2 + 10x + 4\right)(4x+2)\,dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\cos\left(10x^2 + 10x + 4\right)$

   outer ↗ ↖ inner

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(10x^2 + 10x + 4\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(10x^2 + 10x + 4\right)}_{\text{function}} - - - - \rightarrow \underbrace{(4x+2)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(10x^2 + 10x + 4\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 10x^2 + 10x + 4 \\ \Rightarrow \frac{du}{dx} &= 20x + 10 \\ \Rightarrow du &= (20x+10)\,dx \\ \Rightarrow \frac{1}{5}du &= (4x+2)\,dx \end{aligned}$$

3. Analyze in terms of u and du

$$\int \underbrace{\cos\left(10x^2 + 10x + 4\right)}_{\cos(u)}\underbrace{(4x+2)\,dx}_{\frac{1}{5}du} = \int \cos(u)\ \tfrac{1}{5}du = \tfrac{1}{5}\int \cos(u)\,du$$

4. Integrate (in terms of u).

$$\tfrac{1}{5}\int \cos(u)\,du = \tfrac{1}{5}\left[\sin(u)\right] + C = \tfrac{1}{5}\sin(u) + C$$

5. Re-express in terms of the original variable x.

$$\int \cos\left(10x^2 + 10x + 4\right)(4x+2)\,dx = \underbrace{\frac{1}{5}\sin\left(10x^2 + 10x + 4\right) + C}_{\frac{1}{5}\sin(u)+C}$$

i.e., $\int \cos\left(10x^2 + 10x + 4\right)(4x+2)\,dx = \frac{1}{5}\sin\left(10x^2 + 10x + 4\right) + C$

5. **Compute:** $\int_{-1}^{1}\left(4x^3+3x^2+5\right)dx =$

$$\int_{-1}^{1}\underbrace{\left(4x^3+3x^2+5\right)}_{f(x)}dx = \underbrace{\left[4\frac{x^4}{4}+3\frac{x^3}{3}+5x\right]_{-1}^{1}}_{F(x)} = \underbrace{\left[x^4+x^3+5x\right]_{-1}^{1}}_{F(x)} =$$

$$= \underbrace{\left[(1)^4+(1)^3+5(1)\right]}_{F(1)} - \underbrace{\left[(-1)^4+(-1)^3+5(-1)\right]}_{F(-1)} = 7-(-5) = 12$$

i.e., $\int_{-1}^{1}\left(4x^3+3x^2+5\right)dx = 12$

6. **Compute:** $\int_{-1}^{1}\left(3x^3+1\right)^3 x^2\ dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(3x^3+1\right)^3$ (A function raised to a power is *always* a composite function!)

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = \left(3x^3+1\right)$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(3x^3+1\right)}_{\text{function}} - - - - - \rightarrow \underbrace{x^2}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = \left(3x^3+1\right)$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{aligned} u &= 3x^3+1 \\ \Rightarrow \frac{du}{dx} &= 9x^2 \\ \Rightarrow du &= 9x^2\ dx \\ \Rightarrow \frac{1}{9}du &= x^2\ dx \end{aligned}$$

When $x=-1$, $u = 3x^3+1 = 3\left(-1\right)^3+1 = -2$
When $x=1$, $u = 3x^3+1 = 3\left(1\right)^2+1 = 4$

3. Analyze in terms of u and du

$$\int_{x=-1}^{x=1}\underbrace{\left(3x^3+1\right)^3}_{u^3}\underbrace{x^2\ dx}_{\frac{1}{9}du} = \int_{u=-2}^{u=4} u^3\cdot\frac{1}{9}du = \frac{1}{9}\int_{u=-2}^{u=4} u^3 du$$

Don't forget to re-write the limits of integration in terms of u!

4. Integrate (in terms of u).

$$\frac{1}{9}\int_{u=-2}^{u=4} u^3 du = \frac{1}{9}\left[\frac{u^4}{4}\right]_{u=-2}^{u=4} = \left[\frac{u^4}{36}\right]_{u=-2}^{u=4} = \underbrace{\frac{\left(4\right)^4}{36}}_{F(4)} - \underbrace{\frac{\left(-2\right)^4}{36}}_{F(-2)} = \frac{256}{36} - \frac{16}{36} = \frac{240}{36} = \frac{20}{3}$$

i.e., $\int_{x=-1}^{x=1}\left(3x^3+1\right)^3 x^2\ dx = \frac{20}{3}$

7. Compute: $\int \frac{x^2-2x+1}{x^3-3x^2+3x+5}\,dx =$

$$\int \frac{x^2-2x+1}{x^3-3x^2+3x+5}\,dx \underbrace{=}_{\text{re-write}} \int \frac{1}{x^3-3x^2+3x+5}\left(x^2-2x+1\right)\,dx$$

Remark: Note that we have an approximate function/derivative pair, with the "function" in the denominator. This usually indicates that u-substitution is appropriate, with the result being a natural logarithm.

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\frac{1}{x^3-3x^2+3x+5}$ is the same as $\left(x^3-3x^2+3x+5\right)^{-1}$, so it is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = x^3 - 3x^2 + 3x + 5$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(x^3-3x^2+3x+5\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(x^2-2x+1\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = x^3 - 3x^2 + 3x + 5$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{array}{rcl} u & = & x^3 - 3x^2 + 3x + 5 \\ \Rightarrow \frac{du}{dx} & = & 3x^2 - 6x + 3 \\ \Rightarrow du & = & \left(3x^2 - 6x + 3\right) dx \\ \Rightarrow \frac{1}{3}du & = & \left(x^2 - 2x + 1\right) dx \end{array}$$

3. Analyze in terms of u and du

$$\underbrace{\int \frac{1}{x^3-3x^2+3x+5}}_{\frac{1}{u}} \underbrace{\left(x^2-2x+1\right)\,dx}_{\frac{1}{3}du} = \int \frac{1}{u}\frac{1}{3}du = \frac{1}{3}\int \frac{1}{u}du$$

4. Integrate (in terms of u).

$\frac{1}{3}\int \frac{1}{u}du = \frac{1}{3}\ln|u| + C$

5. Re-express in terms of the original variable, x.

$$\int \frac{x^2-2x+1}{x^3-3x^2+3x+5}\,dx = \underbrace{\frac{1}{3}\ln\left|x^3-3x^2+3x+5\right| + C}_{\frac{1}{3}\ln|u|+C}$$

i.e., $\int \frac{x^2-2x+1}{x^3-3x^2+3x+5}dx = \frac{1}{3}\ln\left|x^3-3x^2+3x+5\right| + C$

8. **Compute:** $\int \frac{x^2-2x+1}{(x^3-3x^2+3x+5)^4}\,dx =$

$$\int \frac{x^2-2x+1}{(x^3-3x^2+3x+5)^4}\,dx \underbrace{=}_{\text{re-write}} \int \frac{1}{(x^3-3x^2+3x+5)^4}\,\left(x^2-2x+1\right)\,dx \underbrace{=}_{\text{re-write}} \int \left(x^3-3x^2+3x+5\right)^{-4}\left(x^2-2x+1\right)\,dx$$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes! $\left(x^3-3x^2+3x+5\right)^{-4}$is a function raised to a power.

   Let $u =$ the "inner" of the composite function

   $\Rightarrow u = x^3-3x^2+3x+5$

   b. Is there an (approximate) function/derivative pair?

   Yes! $\underbrace{\left(x^3-3x^2+3x+5\right)}_{\text{function}} - - - - \rightarrow \underbrace{\left(x^2-2x+1\right)}_{\text{deriv}}$

   Let $u =$ the "function" of the function/deriv pair

   $\Rightarrow u = x^3-3x^2+3x+5$

   c. Is the "function" of the function/deriv pair the same as the "inner" of the composite function?

   (i.e., do criteria **a** and **b** suggest the same choice of u ?)

   Yes! $\Rightarrow$ u-substitution is appropriate

2. Compute du

$$\begin{array}{rcl} u & = & x^3-3x^2+3x+5 \\ \Rightarrow \frac{du}{dx} & = & 3x^2-6x+3 \\ \Rightarrow du & = & \left(3x^2-6x+3\right)dx \\ \Rightarrow \frac{1}{3}du & = & \left(x^2-2x+1\right)dx \end{array}$$

3. Analyze in terms of u and du

$$\underbrace{\int \left(x^3-3x^2+3x+5\right)^{-4}}_{u^{-4}} \underbrace{\left(x^2-2x+1\right)\,dx}_{\frac{1}{3}du} = \int u^{-4}\tfrac{1}{3}du = \tfrac{1}{3}\int u^{-4}du$$

4. Integrate (in terms of u).

$$\tfrac{1}{3}\int u^{-4}du = \tfrac{1}{3}\tfrac{u^{-3}}{(-3)} + C = -\tfrac{1}{9}u^{-3} + C$$

5. Re-express in terms of the original variable, x.

$$\int \frac{x^2-2x+1}{(x^3-3x^2+3x+5)^4}\,dx = \underbrace{-\frac{1}{9}\left(x^3-3x^2+3x+5\right)^{-3} + C}_{-\frac{1}{9}u^{-3}+C}$$

$$\text{i.e., } \int \frac{x^2-2x+1}{(x^3-3x^2+3x+5)^4}\,dx = -\tfrac{1}{9}\left(x^3-3x^2+3x+5\right)^{-3} + C$$

9. Compute: $\frac{d}{dx}\left[\ln\left(5x^4+2\sin(x)\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\ln\left(5x^4+2\sin(x)\right)\right]}_{\frac{d}{dx}[\ln(g(x))]} = \underbrace{\frac{1}{5x^4+2\sin(x)}}_{\frac{1}{g(x)}} \cdot \underbrace{\left(20x^3+2\cos(x)\right)}_{g'(x)} = \frac{20x^3+2\cos(x)}{5x^4+2\sin(x)}$$

i.e., $\frac{d}{dx}\left[\ln\left(5x^4+2\sin(x)\right)\right] = \frac{20x^3+2\cos(x)}{5x^4+2\sin(x)}$

10. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2-1}{\cos(x)}}\right)\right] =$ (Use the algebraic properties of natural logarithms to simplify first)

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2-1}{\cos(x)}}\right)\right] \underbrace{=}_{\text{re-write}} \frac{d}{dx}\left[\ln\left[\left(\frac{x^2-1}{\cos(x)}\right)^{\frac{1}{2}}\right]\right] \underbrace{=}_{\text{re-write}} \frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{x^2-1}{\cos(x)}\right)\right] \underbrace{=}_{\text{re-write}} \frac{d}{dx}\left[\frac{1}{2}\left(\ln\left(x^2-1\right)-\ln\left(\cos(x)\right)\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{x^2-1}(2x) - \frac{1}{\cos(x)}\left(-\sin(x)\right)\right) = \frac{1}{2}\left(\frac{2x}{x^2-1}+\tan(x)\right) = \frac{1}{2}\left(\frac{x}{x^2-1}+\frac{1}{2}\tan(x)\right)$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{x^2-1}{\cos(x)}}\right)\right] = \frac{1}{2}\left(\frac{2x}{x^2-1}+\tan(x)\right) = \frac{x}{x^2-1}+\frac{1}{2}\tan(x)$