MTH 4441 MTH 4441 Practice Finale Exam Part #3 - Solutions

Fall 2021

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1. Define - permutation

Let X be a non-empty set. A one to one and onto function $f : X \rightarrow X$ is called a **permutation** of X.

2. **Define -** r**-cycle** (or **cycle**).

Suppose that $x_1, x_2, \ldots, x_r$, with $1 \leq r \leq n$, are distinct elements of $\{1, 2, 3, \ldots, n\}$. The r**-cycle** $(x_1, x_2, \ldots, x_r)$ is the permutation of S_n that maps $x_1 \rightarrow x_2, x_2 \rightarrow x_3, \ldots, x_{r-1} \rightarrow x_r, x_r \rightarrow x_1$, and leaves all other elements fixed.

3. **Prove:** Let $S = \{1, 2, 3, \ldots, n\}$ and let S_n be the set of all permutations $f : S \rightarrow S$. Furthermore, let $\circ$ be the operation of function composition. Then $(S_n, \circ)$ is a group.

pf/

i. The operation $\circ$ on S_X is **closed.**

Let $f, g \in S_X$. Then $f \circ g \in S_X$, since the composition of one to one and onto functions on a set X is also a one to one and onto function on X.

ii. 1_X,the identity function on X, is the identity.

First, note that $1_X \in S_X$, since 1_X is one to one and onto.

Let $f \in S_X$. Then $(1_X \circ f)(x) = 1_X(f(x)) = f(x)$ and $(f \circ 1_X)(x) = f(1_X(x)) = f(x)$.

i.e., $1_X \circ f = f = f \circ 1_X$

iii. Given $f \in S_X$, f **has an inverse.**

Since every permutation $f \in S_X$ is one to one and onto, **every permutation** $f \in S_X$ **has an inverse** $f^{-1} \in S_X$,which has the property that $f^{-1} \circ f = 1_X = f \circ f^{-1}$.

iv. $\circ$ is associative, since the operation of function composition is, in general, associative.

Since $(S_n, \circ)$ satisfies all of the group axioms, it is a group. ■

4. **Define - disjoint cycles**

Two cycles are **disjoint** exactly when they do not "move" (or "act on") the same element.

5. **Define - transposition**

A **transposition** is a 2-cycle. (i.e., a cycle that "moves" or "acts on" exactly two elements).

6. For Exercises 6-7, State two theorems about permutations.

Thm - Let $f \in S_n$.Then there exist disjoint cycles $f_1, f_2, \ldots, f_m \in S_n$, such that $f = f_1 \circ f_2 \circ \ldots \circ f_m$. (i.e., every permutation on $\{1, 2, \ldots, n\}$ can be written as the "product" (actually "composition") of disjoint cycles. The order of these cycles is arbitrary.

7.

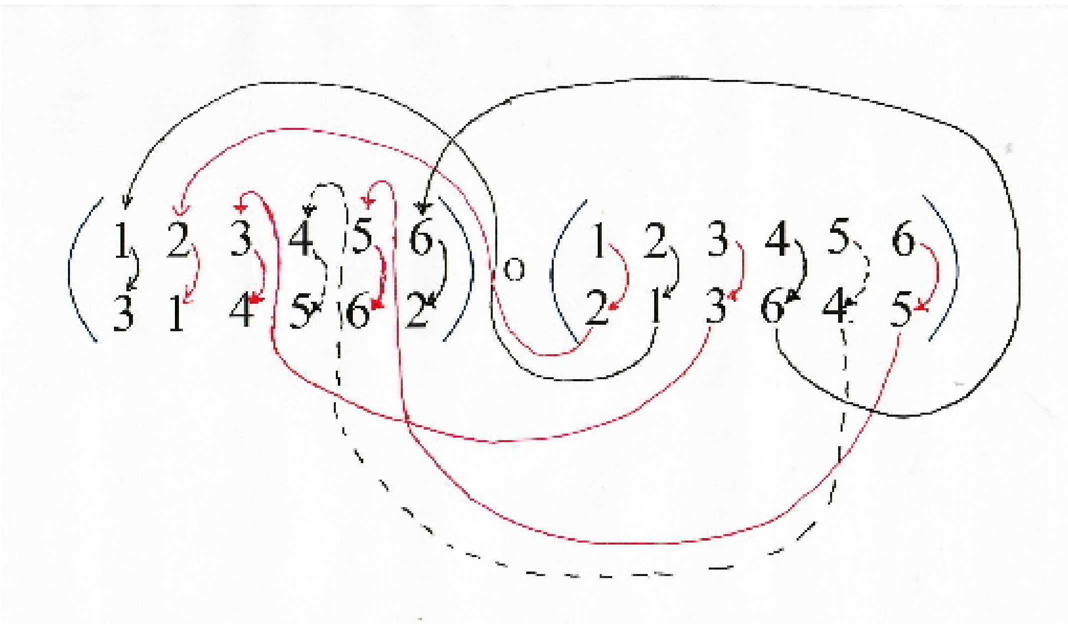
Thm - Every cycle can be expressed as the "product" of transpositions. (in the case of the identity permutation, it can be written as $(1, 2) \circ (1, 2)$)

Thm - A permutation can be expressed as the "product" an even number of transpositions or an odd number of transpositions, but not both. This expression is not unique.

8. Perform the indicated operations in S_6

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 6 & 4 & 5 \end{pmatrix} =$$

Recall: We begin with the permutation on the right.



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 6 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 2 & 5 & 6 \end{pmatrix}$$

Alternatively: We can combine this in one diagram

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 6 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 6 & 4 & 5 \\ 1 & 3 & 4 & 2 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 4 & 2 & 5 & 6 \end{pmatrix}$$

9. Express the permutation as a "product" of disjoint cycles and then as the "product" of transpositions. Classify the permutation as being either **even** or **odd.**

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 6 & 5 & 8 & 7 \end{pmatrix} =$

Starting with 1, note that the permutation maps 1 to 3, 3 to 2, 2 to 4, and 4 back to 1. This yields the cycle $(1,3,2,4)$

We continue with the leftmost element that was not "moved" by cycle $(1,3,2,4)$.

The permutation maps 5 to 6 and 6 back to 5. This yields the cycle $(5,6)$.

We continue with the leftmost element that has not been "moved" by the cycles $(1,3,2,4)$ and $(5,6)$.

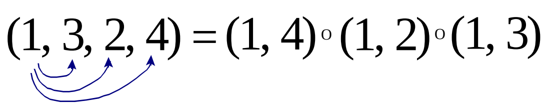
The permutation maps 7 to 8 and 8 back to 7. This yields the cycle $(7,8)$.

Thus, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 6 & 5 & 8 & 7 \end{pmatrix} = (1,3,2,4) \circ (5,6) \circ (7,8)$

The order of the cycles is arbitrary, since the cycles are disjoint.

The cycle $(1,3,2,4)$ can be expressed as the product of transpositions according to the following pattern:

$(1,3,2,4) = (1,4) \circ (1,2) \circ (1,3)$



i.e., $(1,3,2,4) = (1,4) \circ (1,2) \circ (1,3)$ (The order is fixed - it cannot be changed, since the cycles are not disjoint.

Thus, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 6 & 5 & 8 & 7 \end{pmatrix} = \underbrace{(1,4) \circ (1,2) \circ (1,3)}_{=(1,3,2,4)} \circ (5,6) \circ (7,8)$

i.e., $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 6 & 5 & 8 & 7 \end{pmatrix} = (1,4) \circ (1,2) \circ (1,3) \circ (5,6) \circ (7,8)$

Since the permutation can be expressed as the "product" of 5 transpositions, it is an **odd** permutation.

(Continued)

10. Given $(U_5, \odot) = (\{1,2,3,4\}, \odot)$, construct a group of permutations on U_5 that is isomorphic to $(U_5, \odot)$, and exhibit an isomorphism from $(U_5, \odot)$ to this group.

By Cayley's Theorem, every group $(G, *)$ is isomorphic to a group of permutations on G, with the operation being $\circ$ (function composition).

The **standard method** of finding such a group of permutations on G is as follows:

For each element $g \in G$, define the function f_g on G as follows: $f_g(x) = g * x$, $\forall x \in G$

Recall: the group table for $(U_5, \odot) = (\{1,2,3,4\}, \odot)$:

$(U_5, \odot) =$

$\odot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

To construct a group of permutations on U_5 that is isomorphic to $(U_5, \odot)$:
For each element $n \in U_5$, define the function f_n on U_5 as follows: $f_n(x) = n * x$, $\forall x \in G$.

Thus,

$f_1(x) = 1 \cdot x$, for all $x \in U_5$

$f_1(1) = 1 \cdot 1 = 1$

$f_1(2) = 1 \cdot 2 = 2$

$f_1(3) = 1 \cdot 3 = 3$

$f_1(4) = 1 \cdot 4 = 4$

$\Rightarrow f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ $\leftarrow$ The row headed by 1 in the group table

$f_2(x) = 2 \cdot x$, for all $x \in U_5$

$f_2(1) = 2 \cdot 1 = 2$

$f_2(2) = 2 \cdot 2 = 4$

$f_2(3) = 2 \cdot 3 = 1$

$f_2(4) = 2 \cdot 4 = 3$

$\Rightarrow f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ $\leftarrow$ The row headed by 2 in the group table

(Continued)

$f_3(x) = 3 \cdot x$, for all $x \in U_5$

$f_3(1) = 3 \cdot 1 = 3$

$f_3(2) = 3 \cdot 2 = 1$

$f_3(3) = 3 \cdot 3 = 4$

$f_3(4) = 3 \cdot 4 = 2$

$\Rightarrow f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$ $\leftarrow$ The row headed by 3 in the group table

$f_4(x) = 4 \cdot x$, for all $x \in U_5$

$f_4(1) = 4 \cdot 1 = 4$

$f_4(2) = 4 \cdot 2 = 3$

$f_4(3) = 4 \cdot 3 = 2$

$f_4(4) = 4 \cdot 4 = 1$

$\Rightarrow f_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ $\leftarrow$ The row headed by 4 in the group table

Computations will confirm the group table for $(\{f_1, f_2, f_3, f_4\}, \circ)$, shown below:

$(U_5, \odot) =$

$\odot$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

$(\{f_1, f_2, f_3, f_4\}, \circ) =$

$\circ$	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_4	f_1	f_3
f_3	f_3	f_1	f_4	f_2
f_4	f_4	f_3	f_2	f_1

The functions $\phi : (U_5, \odot) \to (\{f_1, f_2, f_3, f_4\}, \circ)$ given by: $\phi(i) = f_i$ transforms the group table for $(U_5, \odot)$ into the group table for $(\{f_1, f_2, f_3, f_4\}, \circ)$

Clearly, the function $\phi : (U_5, \odot) \to (\{f_1, f_2, f_3, f_4\}, \circ)$ given by: $\phi(i) = f_i$ is the isomorphism that we seek.

(Continued)

11. Consider the group $(G, *)$ given in the table below:

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Construct a group of permutations on G that is isomorphic to $(G, *)$, and exhibit an isomorphism from $(G, *)$ to this group.

By Cayley's Theorem, every group $(G, *)$ is isomorphic to a group of permutations on G, with the operation being $\circ$ (function composition).

The **standard method** of finding such a group of permutations on G is as follows:

For each element $g \in G$, define the function f_g on G as follows: $f_g(x) = g * x$, $\forall x \in G$

Recall: the group table for $(G, *) = (\{1, 2, 3, 4\}, \odot)$:

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

To construct a group of permutations on G that is isomorphic to $(G, *)$:
For each element $g \in G$, define the function f_g on G as follows: $f_g(x) = g * x$, $\forall x \in G$.

Thus,

$f_e(x) = e \cdot x$, for all $x \in U_5$

$f_e(e) = e \cdot e = e$

$f_e(a) = e \cdot a = a$

$f_e(b) = e \cdot b = b$

$f_e(c) = e \cdot c = c$

$\Rightarrow f_e = \begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix}$ $\leftarrow$ The row headed by e in the group table

Therefore, f_e is the identity.

In similar fashion, $f_a(x) = a * x$

Thus:

$f_a(e) = a * e = a$

$f_a(a) = a * a = e$

$f_a(b) = a * b = c$

$f_a(c) = a * c = b$

$f_a = \begin{pmatrix} e & a & b & c \\ a & e & c & b \end{pmatrix} \leftarrow$ The row headed by a in the group table

In similar fashion, $f_b(x) = b * x$,

Thus:

$f_b(e) = b * e = b$

$f_b(a) = b * a = c$

$f_b(b) = b * b = e$

$f_b(c) = b * c = a$

$f_b = \begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix} \leftarrow$ The row headed by b in the group table

In similar fashion, $f_c(x) = c * x$

Thus:

$f_c(e) = c * e = c$

$f_c(a) = c * a = b$

$f_c(b) = c * b = a$

$f_c(c) = c * c = e$

$f_c = \begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix} \leftarrow$ The row headed by c in the group table

Some sample computations:

$$f_b \circ f_b = \begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix} \circ \begin{pmatrix} e & a & b & c \\ b & c & e & a \end{pmatrix} = \begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix} = f_e$$

i.e., $f_b \circ f_b = f_e$

$$f_c \circ f_c = \begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix} \circ \begin{pmatrix} e & a & b & c \\ c & b & a & e \end{pmatrix} = \begin{pmatrix} e & a & b & c \\ e & a & b & c \end{pmatrix} = f_e$$

i.e., $f_c \circ f_c = f_e$

The group tables for $(G, *)$ and $(\{f_e, f_a, f_b, f_c\}, \circ)$ are given below:

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\circ$	f_e	f_a	f_b	f_c
f_e	f_e	f_a	f_b	f_c
f_a	f_a	f_e	f_c	f_b
f_b	f_b	f_c	f_e	f_a
f_c	f_c	f_b	f_a	f_e

Key Observation: You may notice that the function $\phi : (G, *) \to (\{f_e, f_a, f_b, f_c\}, \circ)$, given by $\phi(x) = f_x$ transforms the group table for $(G, *)$ into the group table for $(\{f_e, f_a, f_b, f_c\}, \circ)$. Thus, the two groups are isomorphic and ϕ is the isomorphism that we seek.

12. We are given a group $(G, *)$, and an element $x \in G$. Given also that $x^5 = e$ and that $x^3 = e$, prove that $x = e$.

pf/ The key here is to use the facts that:

i) $(x^n)^{-1} = (x^{-1})^n$, where x^{-1} is the inverse of x, and $(x^n)^{-1}$is the inverse of x^n.

and

ii) $e^{-1} = e$

Observe: $e = x^3 \Rightarrow e^{-1} = (x^3)^{-1}$

i.e., $(x^3)^{-1} = e$

Thus, $x^2 = e * x^2 = \left((x^3)^{-1} * x^3\right) * x^2 = (x^3)^{-1} * (x^3 * x^2) = (x^3)^{-1} * x^5 = e * e = e$

i.e., $x^2 = e$

Hence, $(x^2)^{-1} = e$ also.

Thus, $x = e * x = \left((x^2)^{-1} * x^2\right) * x = (x^2)^{-1} * (x^2 * x) = (x^2)^{-1} * x^3 = e * e = e$

i.e., $x = e$ ■