MTH 6610 - History of Math Reading Assignment #5 - Answers

Summer 2018

Pat Rossi Name ______________________

Instructions. Read pages 141 - 181 to find the answers to these questions in your reading.

1. What factors made Alexandria a center of the "Hellenistic World?"

   It had a large harbor (could accommodate 1200 ships) and the city was the commercial junction of trade routes between Asia Africa and Europe.

2. What factors made the Museum (the seat of the Muses) the center of learning of the "Hellenistic World?"

   The leading scholars of the time (scientists, poets, artists, writers) were invited to Alexandria by the Ptolemies and given living accommodations (free board) as long as they wished to stay. They had leisure time to pursue their studies, access to the finest libraries, opportunities to discuss matters with other resident specialists, and were given generous salary stipends.

3. What was the only condition imposed on the lecturers of the Museum?

   They had to give regular lectures.

4. What was the main purpose of the Museum?

   To serve as an institution for research and the pursuit of learning.

5. What observation has been made about a two-hundred year period, during the Museum was at it's height?

   In the history of Mathematics, there is only one other span of about 200 years that can be compared (for productivity) to the period from 300-100 BC, namely the period from Kepler to Gauss (1600-1850).

Remark: The scholars at the Museum were free to choose their course of research. Their choices were not restricted to areas or projects that showed promise of a practical application. And yet, only one other two hundred year period in history compares with the period during which the museum was at its height.

The point of all of this is that a course of study need not have an obvious application in order to be beneficial. Historically, it was often the case that applications sprang from research, rather than vice versa.

Similarly, just because our course of instruction appears (to students and parents) to be lacking any practical value or application, does not mean that it lacks merit.

6. The support of what institution further supported the museum?

   The Alexandrian Library, housing the largest collection of Greek works in existence. Manuscripts were sought throughout the world. Agents commissioned to acquire manuscripts would also borrow (for copying) those manuscripts that they could not obtain outright.

7. What noteworthy mathematical scholars did the Museum produce?

   Euclid, Archimedes, Eratosthenes, Apollonius, Pappus, Claudius Ptolemy, and Diophantus.

8. What is the most noteworthy characteristic of the books, written by Euclid, that are known as "The Elements?"

   The logical arrangement of the theorems and the development of the proofs. He unified a collection of isolated discoveries into a single deductive system based on a set of initial postulates, definitions, and axioms.

9. What noteworthy step did Euclid take, to avoid the problem of "circular reasoning?"

   He established a set of postulates or axioms - statements whose truth was evident, although impossible (or difficult) to prove. (Since their truth is evident, these statements were assumed to be true without the burden of proof.)

10. Name the Postulates of Euclid's "Elements."

    i. A straight line can be drawn from any point to any other point
    ii. A finite straight line can be produced continuously in a line
    iii. A circle may be described with any center and distance
    iv. All right angles are equal to one another
    v. If a straight line falling on any two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

11. Name the Common Notions of Euclid's "Elements."

    i. Things that are equal to the same thing are also equal to one another.
    ii. If equals are added to equals, the wholes are equal.
    iii. If equals are subtracted from equals, the remainders are equal.
    iv. Things that coincide with one another are equal to one another.
    v. The whole is greater than the part.

12. What are some shortcomings of Euclid's work, "The Elements?"

    i. He failed to recognize the necessity of undefined terms, trying instead to define the entire technical vocabulary that he used.
    ii. His set of axioms were inadequate - other prospective axioms were equally necessary for his work.

13. Because of a lack of adequate algebraic symbolism, how did Euclid represent numbers?

    He represented real numbers as line segments.

14. (Referring to the previous question) Consequently, how did Euclid (and the Greeks) solve linear and quadratic equations?

    The lengths of line segments corresponded to the solutions of linear equations and the areas of polygons corresponded to the solutions of quadratic equations.

15. Give detailed explanations as to how Euclid constructed:

    (a) a decagon

    The reasoning on this one is as follows:

    The Greeks knew of the Golden Ratio and the Golden Section. Recall that if a line segment has length a, then the line segment is cut at the Golden Section exactly when the two line segments into which the original segment is cut have lengths x and $a - x$, such that

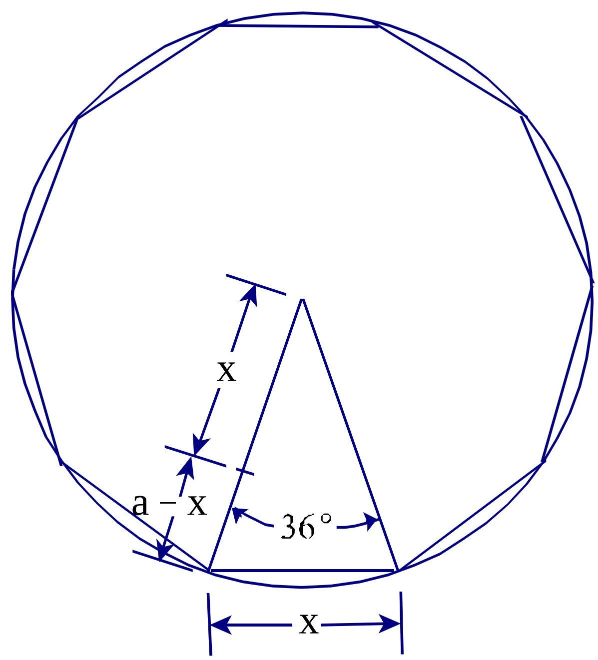
    $$\frac{a}{x} = \frac{x}{a - x}.$$

    (i.e., the longer of the two segments is the mean proportional between the original segment and the shorter of the two segments.)

    Given a line segment of arbitrary length a, the Greeks knew how to cut the line segment at the Golden Section.

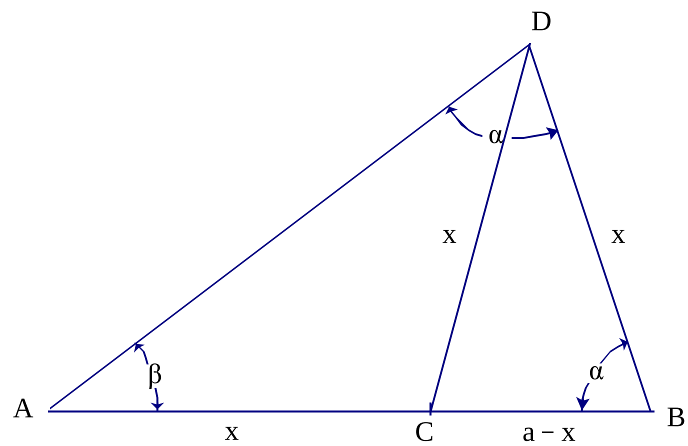
    It turns out that given a decagon, inscribed within the circle of radius a, the sides of the decagon have length x, where $\frac{x}{a}$ is equal to the Golden Ratio. So to construct a regular decagon, they would construct a circle of arbitrary radius a, and then construct a chord of length x, such that $\frac{x}{a}$ is equal to the Golden Ratio. They would repeat the process drawing four more chords, putting one endpoint of the chord at an endpoint of its predecessor.

    Now WHY does this work? Observe that if we draw two radii from the center of a circle to either side of one of the sides of the decagon, we get an isosceles triangle with central angle of measure 36° (see below).



Thus, it suffices to show that given an isosceles triangle with two sides of length a, and smaller side of length x (such that $\frac{x}{a}$ is equal to the Golden Ratio), the angle opposite the side of length x has measure 36^{o}.

So consider the isosceles triangle below, having two sides of length a and remaining side of length x. Let point C be the Golden Section of line segment AB.



(We must show that angle β has measure 36^{o}.)

By the Golden Ratio property,

$$\frac{a}{x} = \frac{x}{a-x}.$$

In terms of lengths of line segments, this becomes:

$$\frac{AB}{AC} = \frac{AC}{CB}.$$

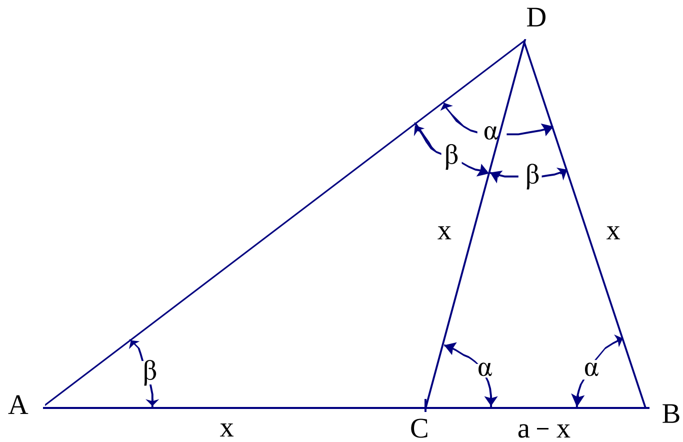
Since $AB = AD = a$ and $AC = DB = x$, we have:

$$\frac{AD}{DB} = \frac{DB}{CB}.$$

This tells us that triangles ADB and DBC are similar (the sides AD and DB of triangle ADB are in proportion to sides DB and CB of triangle DBC, and they have a common angle of measure α.

Consequently, by similar triangles, $\angle DAB = \angle CDB$. We'll call their measure β. Also, line segment DC has length x. Furthermore, triangle ADC is isosceles, with two sides of length x. Hence, their opposite angles are equal (each angle having measure β). Consequently, $\angle ADB = \alpha = 2\beta$. (i.e., $\alpha = 2\beta$).

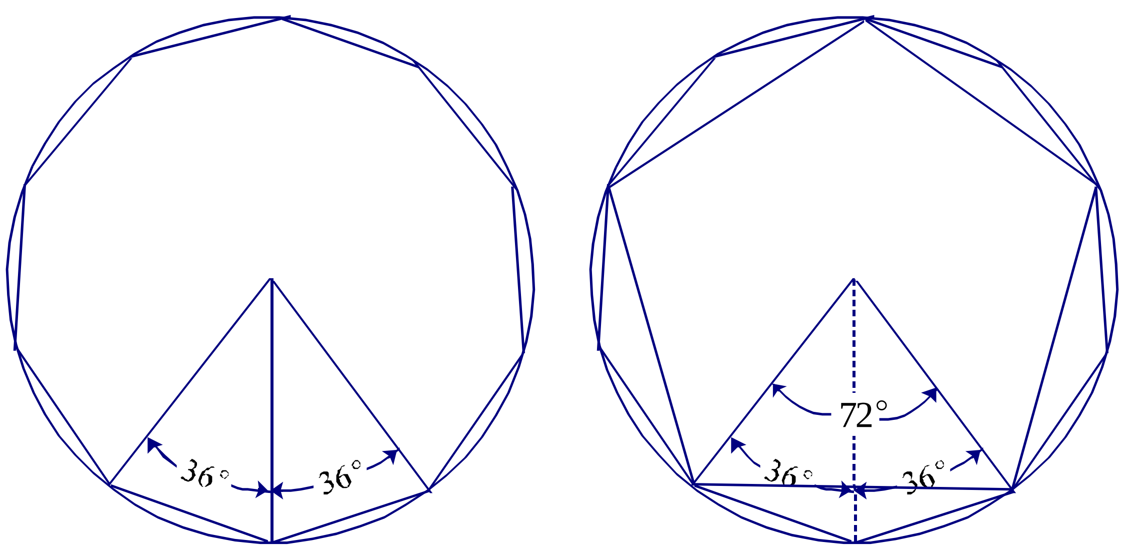
Thus, we have the picture below:



Thus, we have: $180^{\text{o}} = \beta + \alpha + \alpha = \beta + 2\beta + 2\beta = 5\beta$
(i.e., $180^{\text{o}} = 5\beta$)
Hence, $\beta = 36^{\text{o}}$.

(b) a pentagon

OK, this is the *easy* one! We already know how to construct a regular decagon. Recall that the central angle corresponding to a side of the decagon has measure 36^{o}. If we randomly pick a vertex of a decagon and, starting with that vertex, join every other vertex with a line segment, we will have a regular pentagon (see below). (Note that the central angles corresponding to each of these line segments will be 72^{o} $(36^{\text{o}} + 36^{\text{o}})$ and that $5 \cdot 72^{\text{o}} = 360^{\text{o}}$. Hence, we have indeed constructed a regular pentagon.)



16. Why did the Greeks' approach to solving equations retard the progress of algebra for many centuries?

While linear and quadratic equations could be depicted using plane (two-dimensional) geometry, higher degree equations had no such depiction. Thus, higher-degree equations were precluded from consideration.

17. What important results from Number Theory are to be found in "The Elements?"

    i. There are infinitely many primes
    ii. Any integer greater than 1 can be expressed uniquely as the product of primes.
    iii. A formula for computing the sum of numbers in a geometric progression
    iv. A criterion for forming "perfect numbers."

18. Describe the "Euclidean Algorithm." Describe the "step by step process." (Be somewhat specific here.)

    The Euclidean Algorithm is used to compute $\gcd(a, b)$, where $b > 0$.

    We use the Division Algorithm to find q_1, r_1 (with $0 \leq r_1 < b$) such that

    $$a = bq_1 + r_1$$

    Next, we use the Division Algorithm to find q_2, r_2 (with $0 \leq r_2 < r_1$) such that

    $$b = r_1 q_2 + r_2$$

    Inductively, we use the Division Algorithm to find q_{n+1}, r_{n+1} (with $0 \leq r_{n+1} < r_n$) such that

    $$b = r_n q_{n+1} + r_{n+1}$$

    This process produces a strictly decreasing sequence of non-negative integer remainders

    $$r_1 > r_2 > \ldots > r_{n+1} = 0$$

    This sequence must terminate in $r_{n+1} = 0$

    $\gcd(a, b) = r_n$ (the last non-negative remainder).

19. What important property is possessed by any pair of integers that are "relatively prime?"

    Given relatively prime pair of integers a, b, there exist integers x, y such that

    $$ax + by = 1$$

20. State the contents of "Euclid's Lemma."

    If $a|bc$, with $\gcd(a, b) = 1$, $a|c$

21. State the "Fundamental Theorem of Arithmetic."

    Every positive integer $n > 1$ is either prime or can be expressed as a product of primes; this representation is unique up to the order of factorization

22. What major theorem, regarding prime numbers, is attributed to Euclid?

    There are infinitely primes

Also, do the following Homework Exercises:

p. 161 1-6

p. 182 1,2,6,7,10-16