MTH 3311 Test #1 – Solutions

Spring 2021

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Show CLEARLY how you arrive at your answers.

1. Classify the following according to **order** and **linearity.** If an equation is **not linear,** explain why.

(a) $y''' + \sin(x)\, y'' + 2xy = x^2$ **order 3, linear.**

The highest order of derivative of y **is** 3. (y''' is the *third derivative* of y.) Note that y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function.

(b) $y'' + x\sin(y) = \sin(x)$ **order 2, non-linear.**

The highest order of derivative of y is 2. (y'' is the *second derivative* of y.) Since y is the "inner function" of a composite function ($\sin(y)$), the equation is non-linear.

(c) $y''' + 2xy'' + \sqrt{y} = 6x - 6$ **order 3, non-linear.**

The highest order of derivative of y is 3. (y''' is the *third derivative* of y.) Since y is raised to a power other than 1 ($\sqrt{y} = y^{\frac{1}{2}}$), the equation is non-linear.

(d) $\tan(x)\, y''' - 3xy' + 2x^2 y = e^x$ **order 3, linear.**

The highest order of derivative of y **is** 3. (y''' is the *third derivative* of y.) Note that y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function.

(e) $y^{(4)} - y'' + xy = \frac{x}{x^2+1}$ **order 4, linear.**

The highest order of derivative of y **is** 4. ($y^{(4)}$ is the *fourth derivative* of y.) Note that y and its derivatives are all raised to the 1^{st} power, no derivative of y is a "co-factor" of y or any other derivative of y, and neither y nor any of its derivatives are the "inner function" of a composite function.

2. Show that the function $y = c_1e^{-x}+c_2e^{2x}+4x^2+3x+2$ is a solution of the differential equation $y'' - y' - 2y = -8x^2 - 14x + 1$

Observe:

$y = c_1e^{-x} + c_2e^{2x} + 4x^2 + 3x + 2$

$y' = -c_1e^{-x} + 2c_2e^{2x} + 8x + 3$

$y'' = c_1e^{-x} + 4c_2e^{2x} + 8$

Plugging these into the expression $y'' + y' - 2y$, we have:

$$y''-y'-2y = (c_1e^{-x} + 4c_2e^{2x} + 8)-(-c_1e^{-x} + 2c_2e^{2x} + 8x + 3)-2\left(c_1e^{-x} + c_2e^{2x} + 4x^2 + 3x + 2\right)$$

$$= (c_1 + c_1 - 2c_1)\, e^{-x} + (4c_2 - 2c_2 - 2c_2)\, e^{2x} + (8 - 3 - 4) + (-8 - 6)\, x - 8x^2$$
$$= -8x^2 - 14x + 1$$

i.e., $y'' - y' - 2y = -8x^2 - 14x + 1$

Hence, $y = c_1e^{-x} + c_2e^{2x} + 4x^2 + 3x + 2$ is a solution of the equation:

$y'' - y' - 2y = -8x^2 - 14x + 1$

3. Solve: $y' = \frac{e^x e^{-y}}{e^x+1}$; subject to the initial condition $y(0) = 1$ (Assume that $x \geq 0$)

Use the "Separation of Variables" Method

$y' = \frac{e^x e^{-y}}{e^x+1} \Rightarrow \frac{dy}{dx} = \frac{e^x e^{-y}}{e^x+1} \Rightarrow \frac{1}{e^{-y}}\frac{dy}{dx} = \frac{e^x}{e^x+1} \Rightarrow e^y \frac{dy}{dx} = \frac{e^x}{e^x+1} \Rightarrow e^y dy = \frac{e^x}{e^x+1} dx$

i.e., $e^y dy = \frac{e^x}{e^x+1} dx$

The variables are separated, now integrate!

$\int e^y dy = \int \frac{e^x}{e^x+1} dx = \int \frac{1}{e^x+1} e^x dx$

$$\boxed{\begin{array}{lcl} u & = & e^x + 1 \\ \frac{du}{dx} & = & e^x \\ du & = & e^x dx \end{array}}$$

$$\Rightarrow \int e^y dy = \int \underbrace{\frac{1}{e^x+1}}_{\frac{1}{u}} \underbrace{e^x dx}_{du} = \int \frac{1}{u} du = \ln|u| + C = \ln|e^x+1| + C = \ln(e^x+1) + C$$

i.e., $e^y = \ln(e^x+1) + C$ (Eq. 1)

To find C, we refer to the initial condition: $y(0) = 1$

$\Rightarrow e^1 = \ln(e^0+1) + C = \ln(2) + C$

$\Rightarrow e = \ln(2) + C$

$\Rightarrow e - \ln(2) = C$

$\Rightarrow C = e - \ln(2)$

Substituting into Eq. 1, we have:

$\Rightarrow e^y = \ln(e^x+1) + e - \ln(2)$

$$\Rightarrow \boxed{e^y = \ln(e^x+1) + e - \ln(2)}$$

4. Solve: $x\frac{dy}{dx} + 2y = x^3$, with initial condition $y(1) = 0$, using the "Integrating Factor" Method

1. Re-write the equation in the form: $\frac{dy}{dx} + P(x)\,y = Q(x)$

$$\frac{dy}{dx} + \underbrace{2\frac{1}{x}}_{P(x)}\, y = \underbrace{x^2}_{Q(x)}$$

2. Compute the integrating factor:

$$e^{\int P(x)dx} = e^{\int 2\frac{1}{x}dx} = e^{2\int \frac{1}{x}dx} = e^{2\ln|x|} = e^{\ln|x|^2} = e^{\ln\left(x^2\right)} = x^2$$

i.e., $e^{\int P(x)dx} = x^2$

3. Multiply both sides by the integrating factor

$$x^2\frac{dy}{dx} + x^2\left(2\frac{1}{x}\right)y = x^2\left(x^2\right)$$

$$\Rightarrow x^2\frac{dy}{dx} + 2xy = x^4$$

4. Express the left hand side as the derivative of a product

$$\underbrace{x^2}_{1^{\text{st}}}\ \underbrace{\frac{dy}{dx}}_{2^{\text{nd}}\ \text{prime}} + \underbrace{(2x)}_{1^{\text{st}}\ \text{prime}}\ \underbrace{y}_{2^{\text{nd}}} = x^4$$

$$\Rightarrow \frac{d}{dx}\left[x^2 y\right] = x^4$$

5. Integrate both sides w.r.t. x.

$$\int \frac{d}{dx}\left[x^2 y\right]dx = \int x^4 dx$$

$$\Rightarrow x^2 y = \int x^4 dx$$

i.e., $x^2 y = \frac{1}{5}x^5 + C$

6. Solve for y

$$\Rightarrow y = \frac{1}{5}x^3 + Cx^{-2} \quad \text{(Eq. 1)}$$

(Continued)

To find C, we refer to the initial condition: $y(1) = 0$

Plugging $x = 1$, $y = 0$ into Eq. 1, yields:

$0 = \frac{1}{5}(1)^3 + C(1)^{-2} = \frac{1}{5} + C$

i.e., $0 = \frac{1}{5} + C \Rightarrow C = -\frac{1}{5}$

Thus, Eq. 1 becomes: $y = \frac{1}{5}x^3 - \frac{1}{5}x^{-2}$

$\boxed{\text{Our solution is } y = \frac{1}{5}x^3 - \frac{1}{5}x^{-2}}$

5. Determine whether or not the equation is exact. If the equation is exact, solve it.

$(\cos(x) + 6xy^4)\,dx + (12x^2y^3 + 30y^6)\,dy = 0$

This equation can be analyzed as:

$$\underbrace{\left(\cos(x) + 6xy^4\right)}_{M(x,y)}dx + \underbrace{\left(12x^2y^3 + 30y^6\right)}_{N(x,y)}dy = 0$$

By convention, we let $M(x,y)$ be the co-factor of dx and we let $N(x,y)$ be the co-factor of dy.

i.e., $M(x,y) = \cos(x) + 6xy^4$ and $N(x,y) = 12x^2y^3 + 30y^6$

If the Differential equation is **exact,** then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Check: $\frac{\partial M}{\partial y} = 24xy^3 = \frac{\partial N}{\partial x}$

Thus, the equation IS exact, and there exists a function $U(x,y)$ such that the equation $U(x,y) = C$ relates the solution y implicitly as a function of x.

To find $U(x,y)$, we compute the integrals $\int M(x,y)\,dx$ and $\int N(x,y)\,dy$.

$U(x,y) = \int M(x,y)\,dx = \int (\cos(x) + 6xy^4)\,dx = \sin(x) + 3x^2y^4 + f(y)$

$U(x,y) = \int N(x,y)\,dy = \int (12x^2y^3 + 30y^6)\,dy = 3x^2y^4 + \frac{30}{7}y^7 + g(x)$

To find the unknown functions $f(y)$ and $g(x)$, we compare $\int M(x,y)\,dx$ and $\int N(x,y)\,dy$.

$$\begin{array}{ccccccccc} U(x,y) & = & \sin(x) & + & 3x^2y^4 & + & f(y) & + & C \\ & & \uparrow & & \uparrow & & \uparrow & & \\ U(x,y) & = & g(x) & + & 3x^2y^4 & + & \frac{30}{7}y^7 & + & C \end{array}$$

Thus, $f(y) = \frac{30}{7}y^7$ and $g(x) = \sin(x)$, and $U(x,y) = \sin(x) + 3x^2y^4 + \frac{30}{7}y^7 + C$

Our solution $y = y(x)$ is given implicitly by the equation $U(x,y) = C$

$$\boxed{\sin(x) + 3x^2y^4 + \frac{30}{7}y^7 = C}$$

6. Solve: $2xydx+(x^2+y^2)\,dy=0$ using the substitution $v=\frac{y}{x}$. (Assume that $x, y > 0$)

1. Re-express this in the form: $\frac{dy}{dx}=f\left(\frac{y}{x}\right)$

$2xydx+(x^2+y^2)\,dy=0$

$\Rightarrow 2xy+(x^2+y^2)\,\frac{dy}{dx}=0$

$\Rightarrow 2\left(\frac{y}{x}\right)+\left(\frac{x^2}{x^2}+\frac{y^2}{x^2}\right)\frac{dy}{dx}=0$ (We divided both sides by x^2)

$\Rightarrow 2\left(\frac{y}{x}\right)+\left(1+\left(\frac{y}{x}\right)^2\right)\frac{dy}{dx}=0$

$\Rightarrow \left(1+\left(\frac{y}{x}\right)^2\right)\frac{dy}{dx}=-2\left(\frac{y}{x}\right)$

$\Rightarrow \frac{dy}{dx}=-\frac{2\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}$ (Eq. 1)

Thus, we have re-expressed the equation in the form: $\frac{dy}{dx}=f\left(\frac{y}{x}\right)$

2. Let $v=\frac{y}{x}$ (i.e., $y=vx$) $\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

Substituting into Eq. 1, we have:

$\Rightarrow v+x\frac{dv}{dx}=-\frac{2v}{(1+v^2)}$

3. Now Separate!

$\Rightarrow x\frac{dv}{dx}=-\frac{2v}{1+v^2}-v=-\frac{2v}{1+v^2}-v=-\frac{2v}{1+v^2}-\frac{v+v^3}{1+v^2}=-\frac{3v+v^3}{1+v^2}=-\frac{v^3+3v}{v^2+1}$

i.e., $x\frac{dv}{dx}=-\frac{v^3+3v}{v^2+1}$

$\Rightarrow dv=-\frac{v^3+3v}{v^2+1}\frac{1}{x}dx$

$\Rightarrow \frac{v^2+1}{v^3+3v}dv=-\frac{1}{x}dx$

4. Integrate!

$\Rightarrow \int\frac{v^2+1}{v^3+3v}dv=\int-\frac{1}{x}dx$

$$\int\underbrace{\frac{1}{v^3+3v}}_{\frac{1}{u}}\underbrace{\left(v^2+1\right)dv}_{\frac{1}{3}du}=-\int\frac{1}{x}dx$$

$\Rightarrow \frac{1}{3}\int\frac{1}{u}du=-\int\frac{1}{x}dx$

$\Rightarrow \frac{1}{3}\ln|u|=-\ln|x|+C$

$\Rightarrow \ln|u| = -3\ln|x| + C$

$\Rightarrow \ln|v^3 + 3v| = \ln|x|^{-3} + C$

Substituting $\frac{y}{x}$ for v, we have:

$\ln\left|\left(\frac{y}{x}\right)^3 + 3\left(\frac{y}{x}\right)\right| = \ln|x|^{-3} + C$

$\Rightarrow \ln\left(\left(\frac{y}{x}\right)^3 + 3\left(\frac{y}{x}\right)\right) = \ln\left(x^{-3}\right) + C \quad$ (We discarded absolute value bars because $x, y > 0$)

$\Rightarrow e^{\ln\left(\left(\frac{y}{x}\right)^3 + 3\left(\frac{y}{x}\right)\right)} = e^{\ln\left(x^{-3}\right)+C} = e^{\ln\left(x^{-3}\right)}e^C = C_1 e^{\ln\left(x^{-3}\right)} = C_1 x^{-3}$

$\Rightarrow \frac{y^3}{x^3} + 3\left(\frac{y}{x}\right) = C_1 x^{-3}$

$\Rightarrow y^3 + 3x^2y = C_1 \quad$ (We multiplied both sides by x^3)

Our solution y is given implicitly by the equation:

$$\boxed{y^3 + 3x^2y = C_1}$$