MTH 1126 - Test #2 Solutions

Spring, 2004

Pat Rossi Name ______________________

Instructions. Show clearly how you arrive at your answers.

1. Use the facts that $\ln(4) \approx 1.4$ and $\ln(3) \approx 1.1$ to approximate the following:

   (a) $\ln\left(\frac{8}{3}\right) = \ln(8) - \ln(3) = \ln\left(4^{\frac{3}{2}}\right) - \ln(3) = \frac{3}{2}\ln(4) - \ln(3) \approx \frac{3}{2}(1.4) - (1.1) = 1.0$

   ALTERNATIVELY: Note that $\ln(4) = \ln(2^2) = 2\ln(2)$

   i.e., $\ln(4) = 2\ln(2)$

   $\Rightarrow \ln(2) = \frac{1}{2}\ln(4) = \frac{1}{2}(1.4) = 0.7$

   i.e., $\ln(2) = 0.7$

   $\Rightarrow \ln\left(\frac{8}{3}\right) = \ln(8) - \ln(3) = \ln(2^3) - \ln(3) = 3\ln(2) - \ln(3) \approx 3(0.7) - 1.1 = 1.0$

   (b) $\ln(27) = \ln(3^3) = 3\ln(3) \approx 3(1.1) = 3.3$

   (c) $\ln(24) = \ln(8\cdot 3) = \ln(8) + \ln(3) = \ln\left(4^{\frac{3}{2}}\right) + \ln(3) = \frac{3}{2}\ln(4) + \ln(3) \approx \frac{3}{2}(1.4) + 1.1 = 3.2$

   ALTERNATIVELY: Recall that $\ln(2) = 0.7$.

   Hence, $\ln(24) = \ln(2^3\cdot 3) = \ln(2^3) + \ln(3) = 3\ln(2) + \ln(3) \approx 3(0.7) + (1.1) = 3.2$

2. Compute: $\frac{d}{dx}\underbrace{\left[\ln\left(e^x \sin(x)\right)\right]}_{\ln(u)} = \underbrace{\frac{1}{e^x \sin(x)}}_{\frac{1}{u}} \cdot \underbrace{\frac{d}{dx}\left[e^x \sin(x)\right]}_{\frac{du}{dx}} = \frac{1}{e^x \sin(x)} \cdot \underbrace{\left(e^x \sin(x) + \cos(x)\, e^x\right)}_{\text{Product Rule}} =$
$\frac{e^x \sin(x)}{e^x \sin(x)} + \frac{\cos(x) e^x}{e^x \sin(x)} = 1 + \cot(x)$

   ALTERNATIVELY:

   $\frac{d}{dx}\left[\ln\left(e^x \sin(x)\right)\right] = \frac{d}{dx}\left[\ln(e^x) + \ln(\sin(x))\right] = \frac{d}{dx}\left[x + \ln(\sin(x))\right] = 1 + \frac{1}{\sin(x)}\cos(x) = 1 + \cot(x)$

3. Compute: $\frac{d}{dx}\underbrace{\left[e^{\sec(x)}\right]}_{e^u} = \underbrace{e^{\sec(x)}}_{e^u} \cdot \underbrace{\sec(x)\tan(x)}_{\frac{du}{dx}}$

4. Compute: $\int \underbrace{e^{5x^3}}_{e^u}\underbrace{2x^2dx}_{\frac{2}{15}du} = \int e^u \frac{2}{15}du = \frac{2}{15}\int e^u du = \frac{2}{15}e^u + C = \frac{2}{15}e^{5x^3} + C$

$$\boxed{\begin{array}{lrcl} \text{Let} & u & = & 5x^3 \\ \Rightarrow & \frac{du}{dx} & = & 15x^2 \\ \Rightarrow & du & = & 15x^2dx \\ \Rightarrow & \frac{1}{15}du & = & x^2dx \\ \Rightarrow & \frac{2}{15}du & = & 2x^2dx \end{array}}$$

5. Compute: $\int \frac{x^2+2x+2}{2x^3+6x^2+12x-7}dx =$

$$\boxed{\begin{array}{lrcl} \text{Let} & u & = & 2x^3+6x^2+12x-7 \\ \Rightarrow & \frac{du}{dx} & = & 6x^2+12x+12 \\ \Rightarrow & du & = & (6x^2+12x+12)\,dx \\ \Rightarrow & \frac{1}{6}du & = & (x^2+2x+2)\,dx \end{array}}$$

Rewrite as $\int \underbrace{\frac{1}{2x^3+6x^2+12x-7}}_{\frac{1}{u}} \cdot \underbrace{\left(x^2+2x+2\right)dx}_{\frac{1}{6}du} = \int \frac{1}{u}\frac{1}{6}du = \frac{1}{6}\int \frac{1}{u}du =$

$\frac{1}{6}\ln|u| + C = \frac{1}{6}\ln|2x^3+6x^2+12x-7| + C$

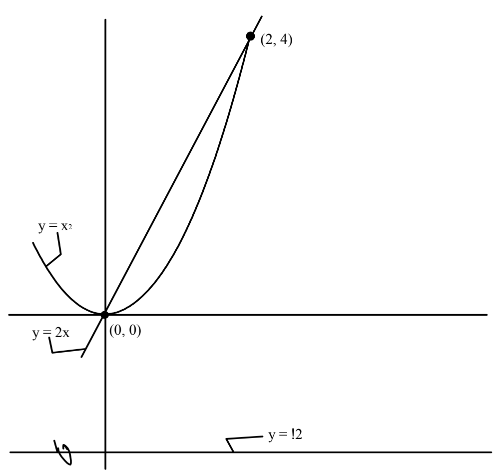
Remark 1 *For problems 6 and 7, draw the rectangle, partition the appropriate interval, build the Riemann Sum, and take the limit.*

6. Use the "Shell Method" to compute the volume of the solid of revolution bounded by the graphs of $y = 2x$ and $y = x^2$ about the axis $y = -2$.

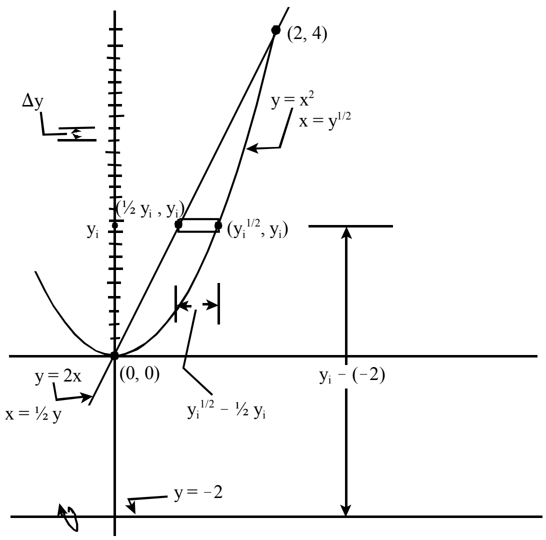
   Sketch the graphs and find the points of intersection.

   To find the points of intersection, we set the y-coordinates equal.

   $y = x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x\,(x-2) = 0 \Rightarrow x = 0$ and $x = 2$.



Sketch a rectangle of width Δy parallel to the axis of revolution, and partition the interval spanned by the rectangles into sub-intervals of width Δy.



Compute the volume of the ith shell

$Vol_i = 2\pi R_i h_i \Delta y = 2\pi\left(y_i - (-2)\right)\left(y_i^{\frac{1}{2}} - \frac{1}{2}y_i\right)\Delta y = 2\pi\left(y_i + 2\right)\left(y_i^{\frac{1}{2}} - \frac{1}{2}y_i\right)\Delta y =$

$2\pi\left(y_i^{\frac{3}{2}} - \frac{1}{2}y_i^2 - y_i + 2y_i^{\frac{1}{2}}\right)\Delta y$

i.e., $Vol_i = 2\pi\left(y_i^{\frac{3}{2}} - \frac{1}{2}y_i^2 - y_i + 2y_i^{\frac{1}{2}}\right)\Delta y$

Approximate the volume of the solid of revolution, by adding the volumes of the shells.

$Vol \approx \sum_{i=1}^{n} 2\pi\left(y_i^{\frac{3}{2}} - \frac{1}{2}y_i^2 - y_i + 2y_i^{\frac{1}{2}}\right)\Delta y$

Let $\Delta y \to 0$

$Vol = \lim_{\Delta y \to 0}\sum_{i=1}^{n} 2\pi\left(y_i^{\frac{3}{2}} - \frac{1}{2}y_i^2 - y_i + 2y_i^{\frac{1}{2}}\right)\Delta y = 2\pi\int_{y=0}^{y=4}\left(y^{\frac{3}{2}} - \frac{1}{2}y^2 - y + 2y^{\frac{1}{2}}\right)dy =$

$2\pi\left[\frac{2}{5}y^{\frac{5}{2}} - \frac{1}{6}y^3 - \frac{1}{2}y^2 + \frac{4}{3}y^{\frac{3}{2}}\right]_0^4 =$

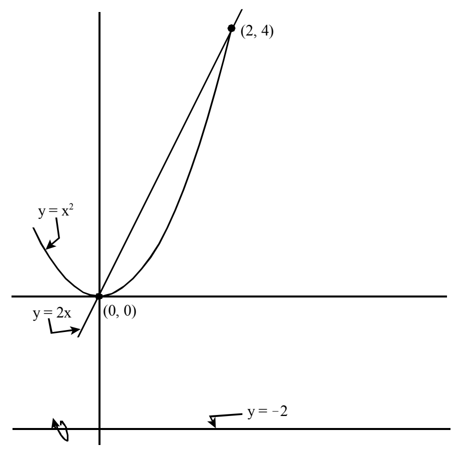
$2\pi\left(\frac{2}{5}(4)^{\frac{5}{2}} - \frac{1}{6}(4)^3 - \frac{1}{2}(4)^2 + \frac{4}{3}(4)^{\frac{3}{2}}\right) - 2\pi\left(\frac{2}{5}(0)^{\frac{5}{2}} - \frac{1}{6}(0)^3 - \frac{1}{2}(0)^2 + \frac{4}{3}(0)^{\frac{3}{2}}\right) = \frac{48}{5}\pi$

7. Use the "Disk Method" to compute the volume of the solid of revolutionbounded by the graphs of $y = 2x$ and $y = x^2$ about the axis $y = -2$.

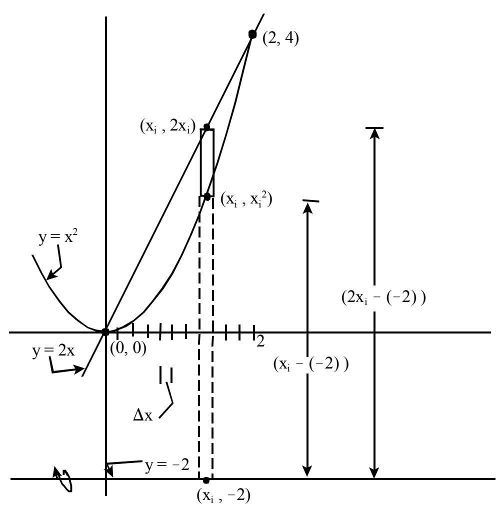
Sketch the graphs and find the points of intersection.

To find the points of intersection, we set the y-coordinates equal.

$y = x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x\,(x-2) = 0 \Rightarrow x = 0$ and $x = 2$.



Draw a rectangle perpendicular to the axis of revolution and partition the interval spanned by the rectangles into sub-intervals of width Δx.



Compute the volume of the i^{th} washer, or donut.

Vol. ith donut = Vol. ith disk − Vol. ith hole $= \pi\left(2x_i - (-2)\right)^2 \Delta x - \pi\left(x_i^2 - (-2)\right)^2 \Delta x =$

$\pi\left(2x_i + 2\right)^2 \Delta x - \pi\left(x_i^2 + 2\right)^2 \Delta x = \pi\left(4x_i^2 + 8x_i + 4\right)\Delta x - \pi\left(x_i^4 + 4x_i^2 + 4\right)\Delta x =$

$\pi\left(-x_i^4+8x_i\right)\Delta x$

i.e., Vol. ith donut $=\pi\left(-x_i^4+8x_i\right)\Delta x$

Approximate the volume of the solid of revolution by adding the volumes of the donuts.

$Vol\approx\sum_{i=1}^{n}\pi\left(-x_i^4+8x_i\right)\Delta x$

Let $\Delta x\rightarrow 0$.

$Vol=\lim_{\Delta x\rightarrow 0}\sum_{i=1}^{n}\pi\left(-x_i^4+8x_i\right)\Delta x=\int_{x=0}^{x=2}\pi\left(-x^4+8x\right)dx=\pi\left[-\frac{1}{5}x^5+4x^2\right]_0^2=$
$\pi\left(-\frac{1}{5}\left(2\right)^5+4\left(2\right)^2\right)-\pi\left(-\frac{1}{5}\left(0\right)^5+4\left(0\right)^2\right)=\frac{48}{5}\pi$