MTH 1126 - Test #1 - 11am Class - Solutions

Spring 2024

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Show CLEARLY how you arrive at your answers

1. Compute: $\frac{d}{dx}\left[e^{\sec\left(5x^3\right)}\right] =$

$$\underbrace{\frac{d}{dx}\left[e^{\sec\left(5x^3\right)}\right]}_{\frac{d}{dx}[e^u]} = \underbrace{e^{\sec\left(5x^3\right)}}_{e^u} \cdot \underbrace{\frac{d}{dx}\left[\sec\left(5x^3\right)\right]}_{\frac{du}{dx}} = e^{\sec\left(5x^3\right)} \cdot \left[\sec\left(5x^3\right)\tan\left(5x^3\right)\cdot\left(15x^2\right)\right]$$

$$= 15x^2\sec\left(5x^3\right)\tan\left(5x^3\right)e^{\sec\left(5x^3\right)}$$

i.e., $\frac{d}{dx}\left[e^{\sec\left(5x^3\right)}\right] = 15x^2\sec\left(5x^3\right)\tan\left(5x^3\right)e^{\sec\left(5x^3\right)}$

2. Compute: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\tan(x)}{4x^3+3x^2}}\right)\right] =$

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\tan(x)}{4x^3+3x^2}}\right)\right] = \frac{d}{dx}\left[\ln\left(\left(\frac{\tan(x)}{4x^3+3x^2}\right)^{\frac{1}{2}}\right)\right] = \frac{d}{dx}\left[\frac{1}{2}\ln\left(\frac{\tan(x)}{4x^3+3x^2}\right)\right]$$

$$= \frac{1}{2}\frac{d}{dx}\left[\left(\ln\left(\tan\left(x\right)\right) - \ln\left(4x^3+3x^2\right)\right)\right]$$

$$= \frac{1}{2}\left(\frac{1}{\tan(x)}\frac{d}{dx}\left[\tan\left(x\right)\right] - \frac{1}{4x^3+3x^2}\frac{d}{dx}\left(4x^3+3x^2\right)\right) = \frac{1}{2}\left(\frac{1}{\tan(x)}\sec^2\left(x\right) - \frac{12x^2+6x}{4x^3+3x^2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{\tan(x)}\sec^2\left(x\right) - \frac{12x+6}{4x^2+3x}\right)$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\tan(x)}{4x^3+3x^2}}\right)\right] = \frac{1}{2}\left(\frac{\sec^2(x)}{\tan(x)} - \frac{12x+6}{4x^2+3x}\right)$

Or: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\tan(x)}{4x^3+3x^2}}\right)\right] = \frac{\sec^2(x)}{2\tan(x)} - \frac{6x+3}{4x^2+3x}$

(Alternative Solution Appears on the Following Page)

Alternative Solution:

$$\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\tan(x)}{4x^3+3x^2}}\right)\right] = \frac{d}{dx}\left[\underbrace{\ln\left[\left(\frac{\tan(x)}{4x^3+3x^2}\right)^{\frac{1}{2}}\right]}_{\ln(u)}\right] = \underbrace{\frac{1}{\left(\frac{\tan(x)}{4x^3+3x^2}\right)^{\frac{1}{2}}}}_{\frac{1}{u}} \cdot \underbrace{\left(\frac{d}{dx}\left[\left(\frac{\tan(x)}{4x^3+3x^2}\right)^{\frac{1}{2}}\right]\right)}_{\frac{du}{dx}}$$

$$= \left(\frac{\tan(x)}{4x^3+3x^2}\right)^{-\frac{1}{2}} \cdot \frac{1}{2}\left(\frac{\tan(x)}{4x^3+3x^2}\right)^{-\frac{1}{2}} \underbrace{\frac{\sec^2(x)(4x^3+3x^2) - (12x^2+6x)\tan(x)}{(4x^3+3x^2)^2}}_{\text{Quotient Rule}}$$

$$= \left(\frac{4x^3+3x^2}{\tan(x)}\right)^{\frac{1}{2}} \cdot \frac{1}{2}\left(\frac{4x^3+3x^2}{\tan(x)}\right)^{\frac{1}{2}} \frac{\sec^2(x)\left(4x^3+3x^2\right) - \left(12x^2+6x\right)\tan(x)}{(4x^3+3x^2)^2}$$

$$= \frac{1}{2}\left(\frac{4x^3+3x^2}{\tan(x)}\right) \frac{\sec^2(x)\left(4x^3+3x^2\right) - \left(12x^2+6x\right)\tan(x)}{(4x^3+3x^2)^2}$$

$$= \frac{1}{2\tan(x)} \frac{\sec^2(x)\left(4x^3+3x^2\right) - \left(12x^2+6x\right)\tan(x)}{(4x^3+3x^2)}$$

$$= \frac{1}{2\tan(x)}\left(\frac{\sec^2(x)\left(4x^3+3x^2\right)}{(4x^3+3x^2)} - \frac{\left(12x^2+6x\right)\tan(x)}{(4x^3+3x^2)}\right)$$

$$= \frac{1}{2\tan(x)}\left(\sec^2(x) - \frac{\left(12x^2+6x\right)\tan(x)}{(4x^3+3x^2)}\right) = \frac{1}{2}\left(\frac{\sec^2(x)}{\tan(x)} - \frac{\left(12x^2+6x\right)\tan(x)}{(4x^3+3x^2)\tan(x)}\right)$$

$$= \frac{1}{2}\left(\frac{\sec^2(x)}{\tan(x)} - \frac{\left(12x^2+6x\right)}{(4x^3+3x^2)}\right) = \frac{1}{2}\left(\frac{\sec^2(x)}{\tan(x)} - \frac{(12x+6)}{(4x^2+3x)}\right)$$

i.e., $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\tan(x)}{4x^3+3x^2}}\right)\right] = \frac{1}{2}\left(\frac{\sec^2(x)}{\tan(x)} - \frac{12x+6}{4x^2+3x}\right)$

Or: $\frac{d}{dx}\left[\ln\left(\sqrt{\frac{\tan(x)}{4x^3+3x^2}}\right)\right] = \frac{\sec^2(x)}{2\tan(x)} - \frac{6x+3}{4x^2+3x}$

3. Compute: $\int e^{\left(3x^6+6x^4\right)}\left(3x^5+4x^3\right)dx =$

(a) 1. Is u-sub appropriate?

a. Is there a composite function?

Yes. $e^{\left(3x^6+6x^4\right)}$

Let $u = 3x^6+6x^4$

b. Is there an "approximate function/derivative pair"?

Yes. $\underbrace{\left(3x^6+6x^4\right)}_{\text{function}} \rightarrow \underbrace{\left(3x^5+4x^3\right)}_{\text{deriv}}$

Let $u = \left(3x^6+6x^4\right)$

2. Compute du

$$\begin{array}{|rrcl|}\hline & u & = & 3x^6+6x^4 \\ \Rightarrow & \frac{du}{dx} & = & 18x^5+24x^3 \\ \Rightarrow & du & = & \left(18x^5+24x^3\right)dx \\ \Rightarrow & \frac{1}{6}du & = & \left(3x^5+4x^3\right)dx \\ \hline\end{array}$$

3. Analyze in terms of u and du.

$$\int \underbrace{e^{\left(3x^6+6x^4\right)}}_{e^u}\underbrace{\left(3x^5+4x^3\right)\,dx}_{\frac{1}{6}du} = \int e^u \tfrac{1}{6}du = \tfrac{1}{6}\int e^u\,du$$

4. Integrate in terms of u

$$\tfrac{1}{6}\int e^u\,du = \tfrac{1}{6}e^u + C$$

5. Re-write in terms of x

$$\int e^{\left(3x^6+6x^4\right)}\left(3x^5+4x^3\right)\,dx = \underbrace{\frac{1}{6}e^{\left(3x^6+6x^4\right)}+C}_{\frac{1}{6}e^u+C}$$

$$\boxed{\text{i.e., } \int e^{\left(3x^6+6x^4\right)}\left(3x^5+4x^3\right)\,dx = \tfrac{1}{6}e^{\left(3x^6+6x^4\right)}+C}$$

4. Compute: $\int \frac{5x^3+3x^2}{(5x^4+4x^3)^5}\,dx = \int \frac{1}{(5x^4+4x^3)^5}\left(5x^3+3x^2\right)\,dx = \int \left(5x^4+4x^3\right)^{-5}\left(5x^3+3x^2\right)\,dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes. $\left(5x^4+4x^3\right)^{-5}$

   Let $u = (5x^4+4x^3)$ i.e., "Let $u =$ the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

   Yes. $\underbrace{\left(5x^4+4x^3\right)}_{\text{function}} \to \underbrace{\left(5x^3+3x^2\right)}_{\text{deriv}}$

   Let $u = (5x^4+4x^3)$ i.e., "Let $u =$ 'the function'"

2. Compute du

$$\begin{array}{rrcl} & u & = & 5x^4+4x^3 \\ \Rightarrow & \frac{du}{dx} & = & 20x^3+12x^2 \\ \Rightarrow & du & = & (20x^3+12x^2)\,dx \\ \Rightarrow & \frac{1}{4}du & = & (5x^3+3x^2)\,dx \end{array}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\int \left(5x^4+4x^3\right)^{-5}}_{u^{-5}} \underbrace{\left(5x^3+3x^2\right)\,dx}_{\frac{1}{4}du} = \int u^{-5}\tfrac{1}{4}du = \tfrac{1}{4}\int u^{-5}\,du$$

4. Integrate in terms of u

$$\tfrac{1}{4}\int u^{-5}\,du = \tfrac{1}{4}\tfrac{u^{-4}}{-4} + C = -\tfrac{1}{16}u^{-4} + C$$

5. Re-write in terms of x

$$\int \frac{5x^3+3x^2}{(5x^4+4x^3)^5}\,dx = \underbrace{-\frac{1}{16}\left(5x^4+4x^3\right)^{-4} + C}_{-\frac{1}{16}u^{-4}+C}$$

$$\boxed{\text{i.e., } \int \frac{5x^3+3x^2}{(5x^4+4x^3)^5}\,dx = -\tfrac{1}{16}\left(5x^4+4x^3\right)^{-4} + C}$$

5. Compute: $\int \frac{2x^3+3x+1}{(x^4+3x^2+2x)}\,dx = \int \frac{1}{(x^4+3x^2+2x)}\,(2x^3+3x+1)\,dx$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes. $\frac{1}{(x^4+3x^2+2x)} = (x^4+3x^2+2x)^{-1}$

   Let $u = (x^4+3x^2+2x)$ i.e., "Let $u =$ the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

   Yes. $\underbrace{(x^4+3x^2+2x)}_{\text{function}} \rightarrow \underbrace{(2x^3+3x+1)}_{\text{deriv}}$

   Let $u = (x^4+3x^2+2x)$ i.e., "Let $u =$ 'the function'"

2. Compute du

$$\boxed{\begin{array}{rrcl} & u & = & x^4+3x^2+2x \\ \Rightarrow & \frac{du}{dx} & = & 4x^3+6x+2 \\ \Rightarrow & du & = & (4x^3+6x+2)\,dx \\ \Rightarrow & \frac{1}{2}du & = & (2x^3+3x+1)\,dx \end{array}}$$

3. Analyze in terms of u and du.

$$\int \underbrace{\int \frac{1}{(x^4+3x^2+2x)}}_{\frac{1}{u}} \underbrace{(2x^3+3x+1)\ dx}_{\frac{1}{6}du} = \int \frac{1}{u}\frac{1}{2}du = \frac{1}{2}\int \frac{1}{u}\,du$$

4. Integrate in terms of u

$\frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

5. Re-write in terms of x

$$\int \frac{2x^3+3x+1}{(x^4+3x^2+2x)}\,dx = \underbrace{\frac{1}{2}\ln\left|x^4+3x^2+2x\right| + C}_{\frac{1}{2}\ln|u|+C}$$

$$\boxed{\text{i.e., } \int \frac{2x^3+3x+1}{(x^4+3x^2+2x)}\,dx = \frac{1}{2}\ln|x^4+3x^2+2x| + C}$$

6. Compute: $\frac{d}{dx}\left[\arcsin\left(\tan\left(x\right)\right)\right] =$

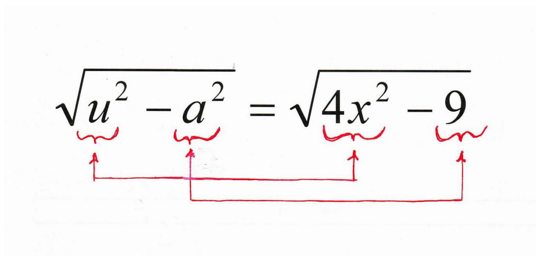
$$\underbrace{\frac{d}{dx}\left[\arcsin\left(\tan\left(x\right)\right)\right]}_{\frac{d}{dx}[\arcsin(u)]} = \underbrace{\frac{1}{\sqrt{1-\left(\tan\left(x\right)\right)^2}}}_{\frac{1}{\sqrt{1-u^2}}} \cdot \underbrace{\sec^2\left(x\right)}_{\frac{du}{dx}} = \frac{\sec^2(x)}{\sqrt{1-(\tan(x))^2}}$$

i.e., $\frac{d}{dx}\left[\arcsin\left(\tan\left(x\right)\right)\right] = \frac{\sec^2(x)}{\sqrt{1-(\tan(x))^2}}$

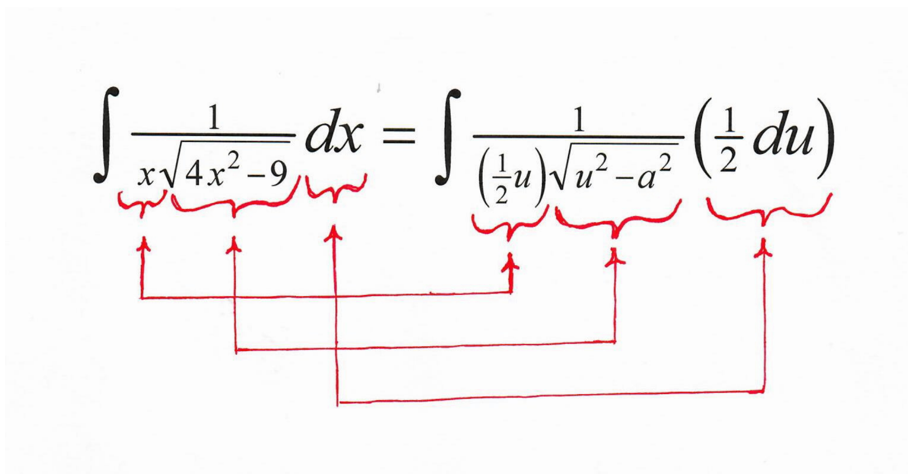
7. Compute: $\int \frac{1}{x\sqrt{4x^2-9}}\,dx =$

This appears to fit the form: $\int \frac{1}{u\sqrt{u^2-a^2}}du$

If our conjecture is correct, then $\sqrt{u^2-a^2} = \sqrt{4x^2-9}$



$\Rightarrow$	$a^2 = 9$
	$a = 3$
$\Rightarrow$	$u^2 = 4x^2$
	$u = 2x$
$\Rightarrow$	$\frac{du}{dx} = 2$
$\Rightarrow$	$du = 2dx$
$\Rightarrow$	$\frac{1}{2}du = dx$
Also:	$u = 2x$
$\Rightarrow$	$\frac{1}{2}u = x$



3. Analyze in terms of u and du.

$$\int \frac{1}{x\sqrt{4x^2-9}}\,dx = \int \frac{1}{x\sqrt{(2x)^2-3^2}}xdx = \int \frac{1}{\left(\frac{1}{2}u\right)\sqrt{u^2-a^2}}\left(\tfrac{1}{2}du\right) = \int \frac{1}{u\sqrt{u^2-a^2}}du$$

4. Integrate

$$\int \frac{1}{u\sqrt{u^2-a^2}}du = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$$

5. Re-express in terms of x

$$\int \frac{1}{x\sqrt{4x^2-9}}\,dx = \underbrace{\frac{1}{3}\operatorname{arcsec}\left(\frac{|2x|}{3}\right) + C}_{\frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)+C}$$

$$\boxed{\int \frac{1}{x\sqrt{4x^2-9}}\,dx = \frac{1}{3}\operatorname{arcsec}\left(\frac{|2x|}{3}\right) + C}$$

8. Compute: $\frac{d}{dx}\left[\cot^{-1}\left(e^x\right)\right] =$

$$\underbrace{\frac{d}{dx}\left[\cot^{-1}\left(e^x\right)\right]}_{\frac{d}{dx}[\cot^{-1}(u)]} = \underbrace{-\frac{1}{1+\left(e^x\right)^2}}_{-\frac{1}{1+u^2}} \cdot \underbrace{e^x}_{\frac{du}{dx}} = -\frac{e^x}{1+e^{2x}}$$

i.e., $\frac{d}{dx}\left[\cot^{-1}\left(e^x\right)\right] = -\frac{e^x}{1+e^{2x}}$

9. Compute: $\int \frac{x^2}{5+9x^6}\,dx = \int \frac{1}{5+9x^6} x^2\,dx$

1. a. Is there a composite function?

   Yes. $\frac{1}{5+9x^6} = (\underbrace{5+9x^6}_{\text{inner}})^{-1}$

   Let $u = 5 + 9x^6$

   Is there an "approximate function/derivative pair"?

   There does not appear to be an "approximate function/derivative pair."

   Proceeding solely on the strength of Part a, we continue, aware of the possibility that u-substitution might not work.

2. Compute du

$$\begin{array}{|lrcl|}\hline & u & = & 5+9x^6 \\ \Rightarrow & \frac{du}{dx} & = & 54x^5 \\ \Rightarrow & du & = & 54x^5 dx \\ \Rightarrow & \frac{1}{54x^3} du & = & x^2\,dx \\ \hline\end{array}$$

3. Analyze in terms of u and du.

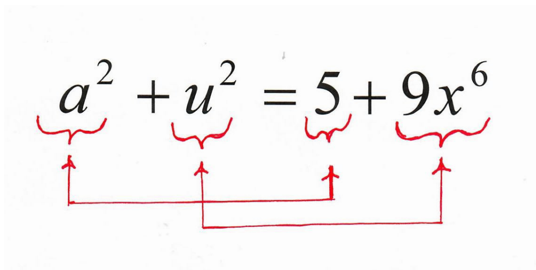
$$\int \underbrace{\frac{1}{5+9x^6}}_{\frac{1}{u}} \cdot \underbrace{x^2\,dx}_{\frac{1}{54x^3}du} =$$

   Since we cannot analyze the integral solely in terms of u and du, u-substitution alone will not work in this case.

   We must try to get our integral to fit a different form. (See Next Page)

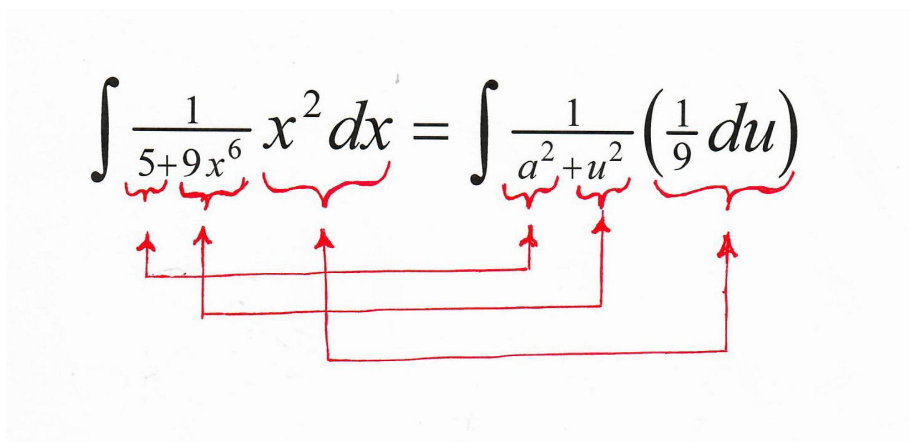
Exercise 9 Continued . . .

$\int \frac{1}{5+9x^6} x^2\, dx$ compare to: $\int \frac{1}{a^2+u^2} du$



If this comparison is correct, then:

$$\begin{aligned} & a^2 = 5 \\ \Rightarrow\ & a = \sqrt{5} \\ & u^2 = 9x^6 \\ \Rightarrow\ & u = 3x^3 \\ \Rightarrow\ & \tfrac{du}{dx} = 9x^2 \\ \Rightarrow\ & du = 9x^2 dx \\ \Rightarrow\ & \tfrac{1}{9} du = x^2 dx \end{aligned}$$



3. Now analyze the integral in terms of u and du.

$$\int \frac{1}{5+9x^6} x^2\, dx = \int \frac{1}{\left(\sqrt{5}\right)^2 + (3x^3)^2} x^2\, dx = \int \frac{1}{a^2+u^2} \frac{1}{9} du = \frac{1}{9} \int \frac{1}{a^2+u^2} du$$

4. Integrate:

$$\frac{1}{9} \int \frac{1}{a^2+u^2} du = \frac{1}{9} \cdot \frac{1}{a} \arctan\left(\frac{u}{a}\right) + C = \frac{1}{9} \cdot \frac{1}{\sqrt{5}} \arctan\left(\frac{3x^3}{\sqrt{5}}\right) + C = \frac{1}{9\sqrt{5}} \arctan\left(\frac{3x^3}{\sqrt{5}}\right) + C$$

i.e., $\int \frac{1}{5+9x^6} x^2\, dx = \frac{1}{9\sqrt{5}} \arctan\left(\frac{3x^3}{\sqrt{5}}\right) + C$

10. $z = \tan\left(\operatorname{arcsec}\left(\frac{3x}{2}\right)\right)$ Re-write this equation as an equivalent algebraic equation.

Let $w = \operatorname{arcsec}\left(\frac{3x}{2}\right)$

Then "w is the angle whose secant is $\frac{3x}{2}$."

i.e., $\sec(w) = \frac{3x}{2}$

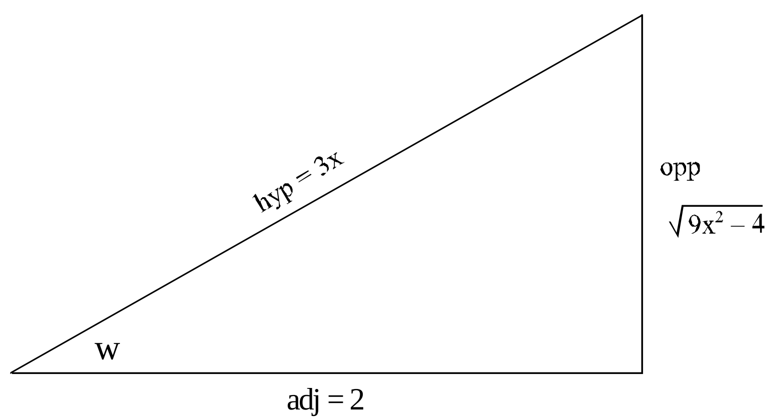
Draw a right triangle that depicts this relationship.

i.e., $\sec(w) = \frac{3x}{2} = \frac{\text{hyp}}{\text{adj}}$

$opp^2 = hyp^2 - adj^2 = (3x)^2 - 2^2 = 9x^2 - 4$

i.e., $opp^2 = 9x^2 - 4$

$\Rightarrow opp = \sqrt{9x^2 - 4}$



We want $z = \tan\left(\operatorname{arcsec}\left(\frac{3x}{2}\right)\right)$

But since $w = \operatorname{arcsec}\left(\frac{3x}{2}\right)$,

$\Rightarrow z = \tan(w)$

$\Rightarrow z = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{9x^2-4}}{2}$

i.e., $z = \frac{\sqrt{9x^2-4}}{2}$

Extra: Wow! 10 points (All or nothing)

Compute: $\int \frac{1}{x\sqrt{16x^4-9}}\,dx =$

1. Is u-sub appropriate?

   a. Is there a composite function?

   Yes. $\frac{1}{\sqrt{16x^4-9}} = (16x^4-9)^{-\frac{1}{2}}$

   Let $u = 16x^4 - 9$ i.e., "Let $u =$ the 'inner' function"

   b. Is there an "approximate function/derivative pair"?

   ???. I sure don't see one! I don't see the derivative of $16x^4 - 9$ anywhere.

   We have $\frac{1}{u}$, but we don't have a constant multiple of du anywhere!

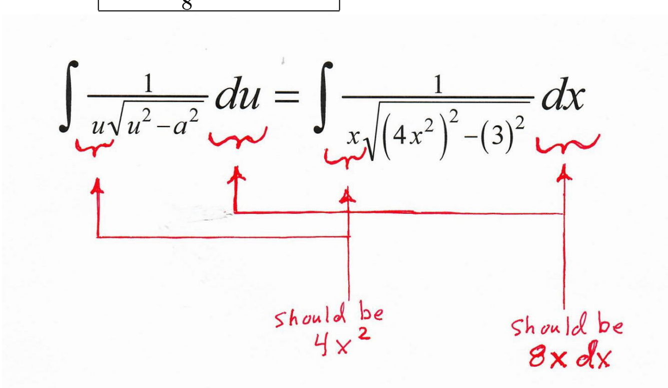
   Let's go to "Plan B"

   The expression $\int \frac{1}{x\sqrt{16x^4-9}}\,dx$ resembles the form: $\int \frac{1}{u\sqrt{u^2-a^2}}\,du$.

   Can we make our integral fit that form?

   If so, then $u^2 = 16x^4$ and $a^2 = 9$

$$\begin{array}{ll} & a^2 = 9 \\ \Rightarrow & a = 3 \\ & u^2 = 16x^4 \\ \Rightarrow & u = 4x^2 \\ \Rightarrow & \frac{du}{dx} = 8x \\ \Rightarrow & du = 8x\,dx \\ \Rightarrow & \frac{1}{8}du = x\,dx \end{array}$$



Note, from the diagram above, that both the denominator and the numerator are lacking a factor of x. (x should be $4x^2$ and dx should be $8x\,dx$)

No Problem – we just have to multiply both numerator and denominator by x. This yields:

$$\int \frac{1}{x\sqrt{16x^4-9}}\,dx = \int \underbrace{\frac{x}{x}}_{=1} \frac{1}{x\sqrt{16x^4-9}}\,dx = \int \frac{1}{x^2\sqrt{16x^4-9}}\,xdx = \int \frac{1}{\left(\frac{1}{4}u\right)\sqrt{u^2-a^2}}\,\frac{1}{8}du = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-a^2}}du$$

$$= \frac{1}{2}\frac{1}{a}\,\text{arcsec}\left(\frac{|u|}{a}\right) + C = \frac{1}{2}\frac{1}{3}\,\text{arcsec}\left(\frac{\left|4x^2\right|}{3}\right) + C = \frac{1}{6}\,\text{arcsec}\left(\frac{\left|4x^2\right|}{3}\right) + C$$

$$\boxed{\text{i.e., } \int \frac{1}{x\sqrt{16x^4-9}}\,dx = \frac{1}{6}\,\text{arcsec}\left(\frac{\left|4x^2\right|}{3}\right) + C}$$