MTH 1125 Test #3

Summer 2023

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Instructions. Show CLEARLY how you arrive at your answers.

1. $f(x) = 2x^3+3x^2-36x+8$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

   i. Compute $f'(x)$ and find critical numbers

   $f'(x) = 6x^2 + 6x - 36$

   a. "Type a" $(f'(c) = 0)$

   Set $f'(x) = 6x^2 + 6x - 36 = 0$

   $\Rightarrow 6x^2 + 6x - 36 = 0$

   $\Rightarrow x^2 + x - 6 = 0$

   $\Rightarrow (x+3)(x-2) = 0$

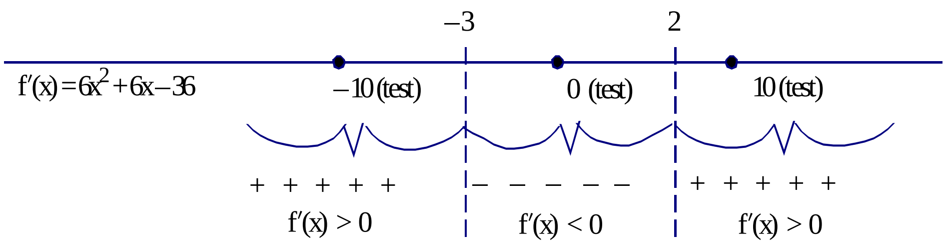
   $\Rightarrow x = -3;\ x = 2$ critical numbers

   b. "Type b" $(f'(c)$ undefined$)$

   There are none.

   ii. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

   iii. From each interval select a "test point" to plug into $f'(x)$



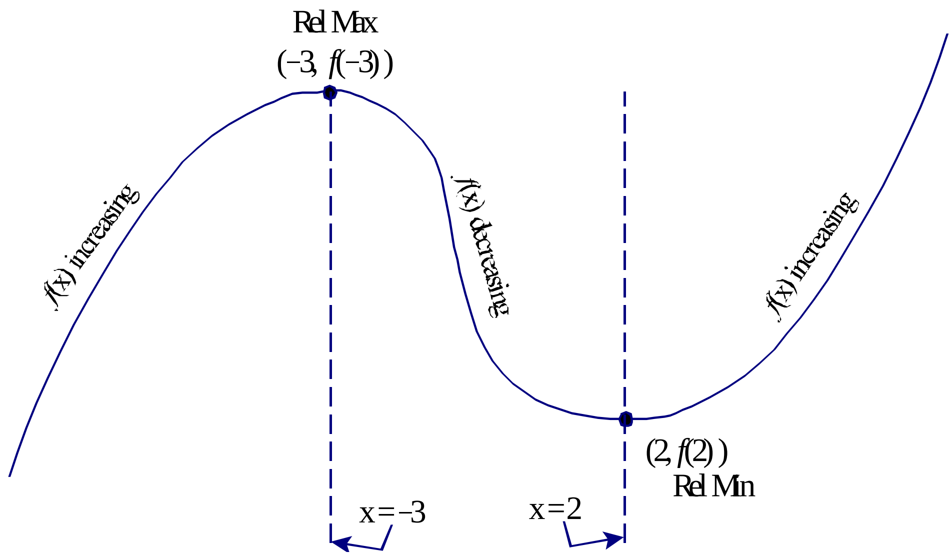
$f(x)$ is **increasing** on the intervals $(-\infty, -3)$ and $(2, \infty)$

(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the interval $(-3, 2)$

(Because $f'(x)$ is negative on this interval)

iv. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



$f(x)$ is **increasing** on the intervals $(-\infty, -3)$ and $(2, \infty)$

$f(x)$ is **decreasing** on the interval $(-3, 2)$

$(-3, f(-3)) = (-3, 89)$ Relative Max

$(2, f(3)) = (2, -36)$ Relative Min

2. $f(x) = x^4 - 4x^3 - 48x^2 + 6x - 6$ Determine the intervals on which $f(x)$ is Concave up/Concave down and identify all points of inflection.

   i. Compute $f''(x)$ and find possible points of inflection

   $f'(x) = 4x^3 - 12x^2 - 96x + 6$

   $f''(x) = 12x^2 - 24x - 96$

   a. "Type a" $(f''(c) = 0)$

   Set $f''(x) = 12x^2 - 24x - 96 = 0$

   $\Rightarrow 12x^2 - 24x - 96 = 0$

   $\Rightarrow x^2 - 2x - 8 = 0$

   $\Rightarrow (x+2)(x-4) = 0$

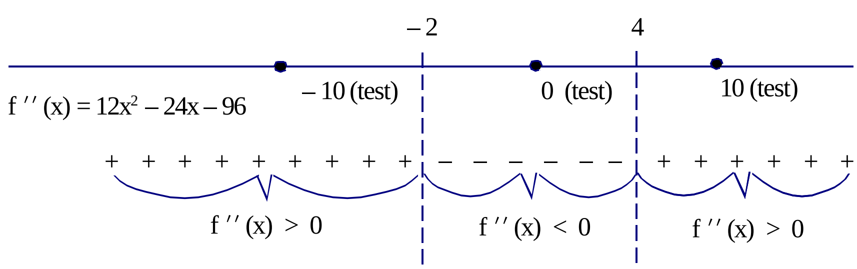
   $\Rightarrow x = -2$; and $x = 4$ possible points of inflection

   b. "Type b" ($f''(c)$ undefined)

   There are none.

   ii. Draw a sign graph of $f''(x)$, using the possible points of inflection to partition the x-axis

   iii. From each interval select a "test point" to plug into $f''(x)$



   $f(x)$ is **concave up** on the intervals $(-\infty, -2)$ and $(4, \infty)$

   (Because $f''(x) > 0$ on these intervals)

   $f(x)$ is **concave down** on the interval $(-2, 4)$

   (Because $f''(x) < 0$ on this interval)

   Since $f(x)$ changes concavity at $x = -2$ and $x = 4$, the points:

   $(-2, f(-2)) = (-2, -162)$

   and

   $(4, f(4)) = (4, -750)$ *are* points of inflection.

3. $f(x) = x^3 + 6x^2 - 6$ on the interval $[-3, 2]$. Find the Absolute Maximum and Absolute Minimum values (if they exist).

Since $f(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[-3, 2]$, we can use the Absolute Max/Min Value Test

i. Compute $f'(x)$ and find the critical numbers

$f'(x) = 3x^2 + 12x$

a. "Type a" $(f'(c) = 0)$

Set $f'(x) = 3x^2 + 12x = 0$

$\Rightarrow 3x^2 + 12x = 0$

$\Rightarrow 3x(x+4) = 0$

$\Rightarrow x = 0$; $x = -4$ "type a" crit. numbers

Since $-4 \notin [-3, 2]$, we discard $x = -4$ as a critical number.

b. "Type b" ($f'(c)$ undefined)

There are none.

ii. Plug critical numbers and endpoints into the original function.

$f(-3) = (-3)^3 + 6(-3)^2 - 6 = 21$

$f(0) = (0)^3 + 6(0)^2 - 6 = -6 \leftarrow$ Abs Min Value

$f(2) = (2)^3 + 6(2)^2 - 6 = 26 \leftarrow$ Abs Max Value

Abs Max Value $= 26$ (attained at $x = 2$) Abs Min Value $= -6$ (attained at $x = 0$)

4. $f(x) = x^{\frac{8}{3}} - 16x^{\frac{2}{3}} - 2$ Determine the intervals on which $f(x)$ is increasing/decreasing and identify all relative maximums and minimums.

i. Compute $f'(x)$ and find critical numbers$\frac{8}{3}x^{\frac{5}{3}} - \frac{32}{3}x^{-\frac{1}{3}}$

$$f'(x) = \frac{8}{3}x^{\frac{5}{3}} - \frac{32}{3}x^{-\frac{1}{3}} = \frac{8x^{\frac{5}{3}}}{3} - \frac{32}{3x^{\frac{1}{3}}} = \frac{8x^{\frac{5}{3}}}{3}\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}} - \frac{32}{3x^{\frac{1}{3}}} = \frac{8x^2}{3x^{\frac{1}{3}}} - \frac{32}{3x^{\frac{1}{3}}} = \frac{8x^2-32}{3x^{\frac{1}{3}}}$$

i.e., $f'(x) = \frac{8x^2-32}{3x^{\frac{1}{3}}}$

a. "Type a" $(f'(c) = 0)$

Set $f'(x) = \frac{8x^2-32}{3x^{\frac{1}{3}}} = 0$

$\Rightarrow 8x^2 - 32 = 0$

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow (x+2)(x-2) = 0$

$\Rightarrow x = -2;\ x = 2$ ("type a" critical numbers)

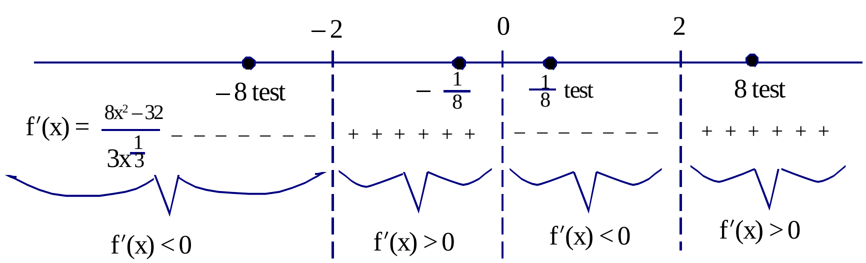
b. "Type b" ($f'(c)$ undefined)

$f'(x)$ is undefined when $3x^{\frac{1}{3}} = 0$

$\Rightarrow x = 0$ ("type a" critical numbers)

ii. Draw a sign graph of $f'(x)$, using the critical numbers to partition the x-axis

iii. From each interval select a "test point" to plug into $f'(x)$



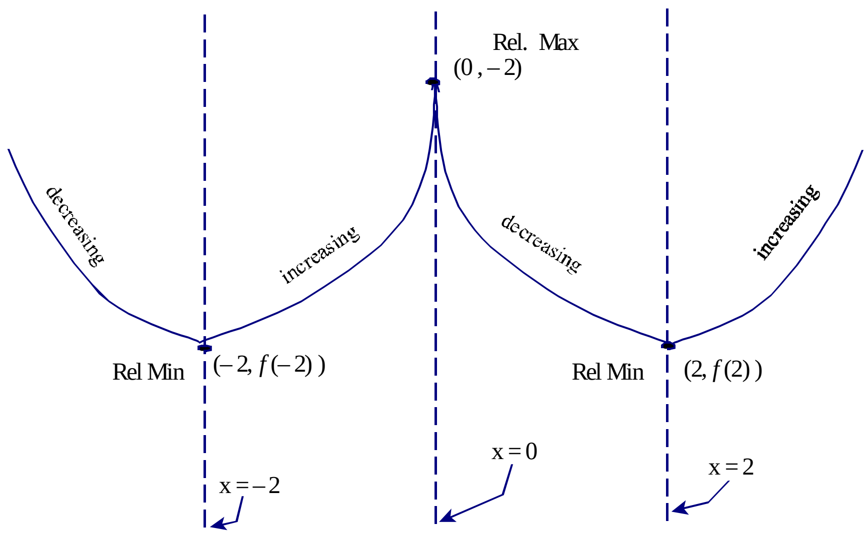
$f(x)$ is **increasing** on the intervals $(-2, 0)$ and $(2, \infty)$

(Because $f'(x)$ is positive on these intervals)

$f(x)$ is **decreasing** on the intervals $(-\infty, -2)$ and $(0, 2)$

(Because $f'(x)$ is negative on these intervals)

iv. To find the relative maxes and mins, sketch a rough graph of $f(x)$.



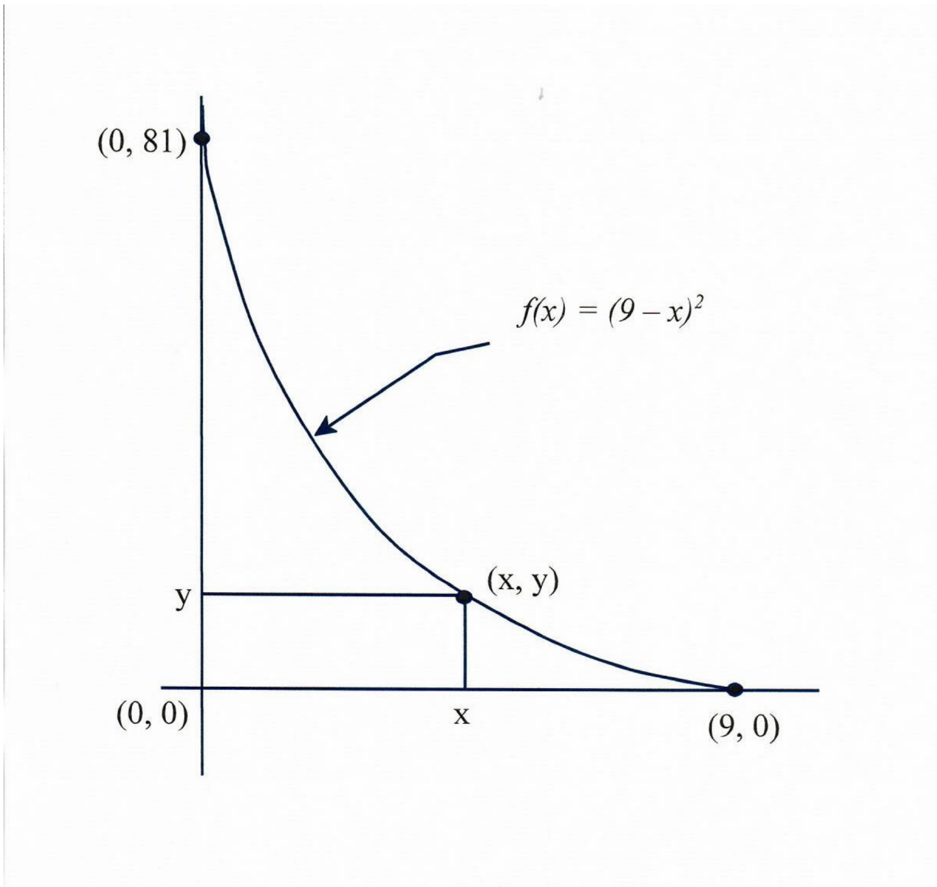
$f(x)$ is **increasing** on the intervals $(-2, 0)$ and $(2, \infty)$

$f(x)$ is **decreasing** on the intervals $(-\infty, -2)$ and $(0, 2)$

Relative Min: $(-2, f(-2))$ and $(2, f(2))$

Relative Max: $(0, f(0)) = (0, -2)$

5. A rectangle is to be constructed such that one side lies on the positive y-axis, an adjacent side lies on the positive x-axis, and the vertex in between is the origin. If the opposite vertex lies on the graph of $f(x) = (9-x)^2$, what should the value of x be such that the area of the rectangle is as large as possible?



i. Determine the quantity to be maximized/minimized - give it a name,

   Maximize the overall area of the pen, $A = xy$

ii. Express A as a function of *one* variable.

   (Refer to a restriction stated in the problem to do this)

   Restriction: y is the value of $f(x)$ at the point $(x, f(x)) = \left(x, (9-x)^2\right)$

   Thus, $A = xy = x(9-x)^2$

   i.e., $A(x) = x(9-x)^2 = x^3 - 18x^2 + 81x$

   i.e., $A(x) = x^3 - 18x^2 + 81x$

iii. Determine the restrictions on y

   $0 \leq x \leq 9$

iv. Maximize/minimize, using the techniques of calculus.

Observe: $A(x)$ is [1]continuous (it's a polynomial) on the [2]closed, [3]finite interval $[0, 9]$.

Therefore, we can use the Absolute Max/Min Value Test

1. Compute $A'(x)$ and find the critical numbers

   $A'(x) = 3x^2 - 36x + 81 = 3(x^2 - 12x + 27) = 3(x-3)(x-9)$

   $A'(x) = 3(x-3)(x-9)$

   a. "Type a" $(A'(c) = 0)$

      $A'(x) = 3(x-3)(x-9) = 0$

      $\Rightarrow x = 3,\ x = 9$ critical numbers

   b. "Type b" ($A'(c)$ is undefined)

      There are none.

2. Plug the critical numbers and endpoints into the *original function.*

   $A(0) = (0)(9-(0))^2 = 0 \leftarrow$ Abs Min Value

   $A(3) = (3)(9-(3))^2 = 108 \leftarrow$ Abs Max Value

   $A(9) = (9)(9-(9))^2 = 0 \leftarrow$ Abs Min Value

5. Make sure that we've solved the original question (problem).

   "What should the value of x be such that the area of the rectangle is as large as possible?"

   We have the Abs Max Area when $x = 3$

   $\boxed{x = 3}$